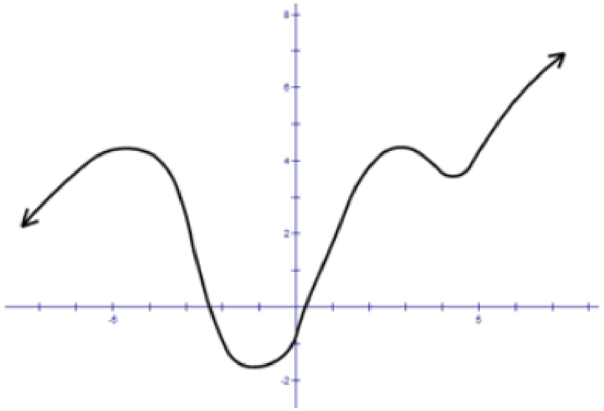
Polynomial Approximations

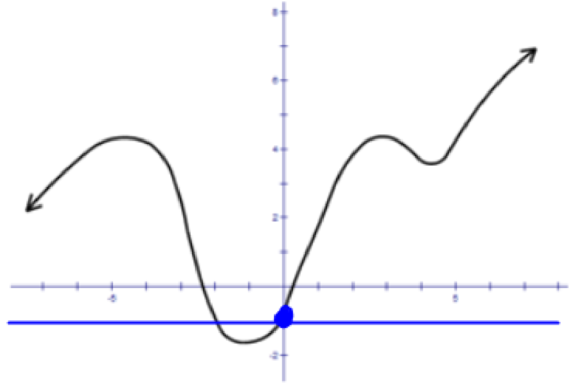
- In this next section of the course, we will focus on using **polynomial functions** to **approximate** other function types.



- Let's consider any arbitrary function $f(x)$, represented by the graph. For now, let's approximate this function where $x = 0$. By doing this, we say that the approximating polynomial is **centered at 0.**

Building the approximating polynomial:

- The most basic approximation would be estimating this function using a constant.



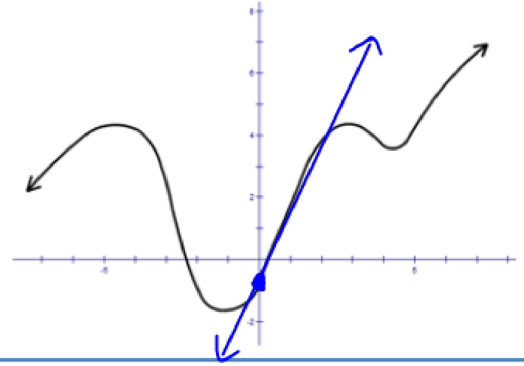
$$f(x) \approx P(x) = f(0)$$

Guarantees the point @ $x=0$

- Linear function approximation. $b + mx$

$$f(x) \approx P(x) = f(0) + f'(0)x$$

Guarantees the slope @ $x=0$

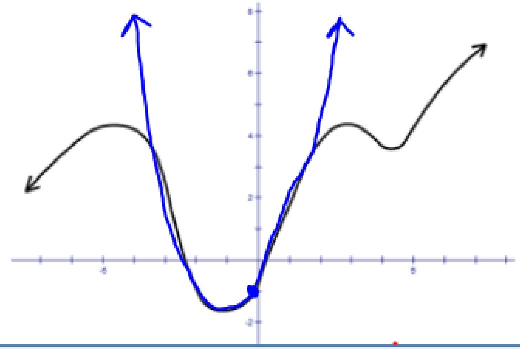


-Quadratic function approximation. $a + bx + ax^2$

$$f(x) \approx P(x) = f(0) + f'(0)x + f''(0) \cdot ax^2$$

$$= f(0) + f'(0)x + f''(0)\frac{x^2}{2}$$

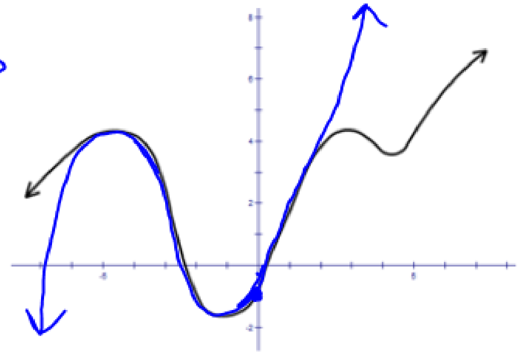
Guarantees concavity @ $x=0$



-Cubic function approximation.

$$f(x) \approx P(x) = f(0) + f'(0)x + f''(0) \cdot \frac{x^2}{2} + f'''(0) \cdot ax^3$$

$$= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6}$$



0 Degree Polynomial	$P_0(x) = f(0)$
1st Degree Polynomial	$P_1(x) = f(0) + f'(0) \cdot \frac{x^1}{1!}$
2nd Degree Polynomial	$P_2(x) = f(0) + f'(0) \cdot \frac{x^1}{1!} + f''(0) \cdot \frac{x^2}{2!}$
3rd Degree Polynomial	
4th Degree Polynomial	
⋮	⋮
nth Degree Polynomial	$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(n)}(0) \cdot \frac{x^n}{n!}$

- Using **<u>higher degrees</u>** of polynomial approximations results in **<u>more accurate approximations</u>** of any arbitrary or specific function.
- If we continue this process **<u>to infinity</u>**, the approximating polynomial becomes an **<u>exact representation of the given function.</u>**

Maclaurin Polynomials Taylor Polynomonmial

A polynomial approximation for a given function, $f(x)$, **centered at $x = 0$** is called the **n^{th} Maclaurin polynomial for f**. The n^{th} Maclaurin polynomial is given by

$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}$$

Example 1: Find the 3rd degree and the nth degree Maclaurin Polynomials for $f(x) = e^x$.

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$f'''(x) = e^x$ $\quad f'''(0) = 1$

$$P_3(x) = 1 + 1 \cdot x + 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^3}{3!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}$$

$P_3(0.1) = 1.1051\bar{6}$

$f(0.1) = e^{0.1} = 1.105170981$

Example 2: Find the 4th degree Maclaurin Polynomial for $f(x) = \sqrt{1+x}$. Use the Maclaurin polynomial to approximate $f(0.1)$.

$f(x) = (1+x)^{1/2}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\qquad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\qquad f'''(0) = \frac{3}{8}$

$f^{(4)}(x) = \frac{-15}{16}(1+x)^{-7/2}$ $\qquad f^{(4)}(0) = \frac{-15}{16} \longrightarrow f^{(n)}(0) = (-1)^{n+1} \cdot \frac{\quad}{2^n}$

$$P_4(x) = 1 + \frac{1}{2}\cdot x + -\frac{1}{4}\cdot\frac{x^2}{2!} + \frac{3}{8}\cdot\frac{x^3}{3!} + \frac{-15}{16}\cdot\frac{x^4}{4!}$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$$

$$P_4(0.1) = 1.048808594$$

$$f(0.1) = \sqrt{1.1} = 1.048808848$$

Example 3: Find the 5th degree Maclaurin Polynomials for $f(x) = \ln(x+1)$. Use the 5th degree Maclaurin polynomial to approximate $f(0.1), f(0.2), \text{and } f(0.5)$.

$f(x) = \ln(x+1)$ $\qquad f(0) = 0$

$f'(x) = (x+1)^{-1}$ $\qquad f'(0) = 1$

$f''(x) = -(x+1)^{-2}$ $\qquad f''(0) = -1$

$f'''(x) = 2(x+1)^{-3}$ $\qquad f'''(0) = 2$

$f^{(4)}(x) = -6(x+1)^{-4}$ $\qquad f^{(4)}(0) = -6$

$f^{(5)}(x) = 24(x+1)^{-5}$ $\qquad f^{(5)}(0) = 24$

$$P_5(x) = 0 + 1\cdot x + -1\cdot\frac{x^2}{2!} + 2\cdot\frac{x^3}{3!} + -6\cdot\frac{x^4}{4!} + 24\cdot\frac{x^5}{5!}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots + (-1)^{n+1}\cdot\frac{x^n}{n}$$

$P_5(0.1) = 0.095310\overline{3}$ $\qquad f(0.1) = 0.0953101798\,04$

$P_5(0.2) = 0.182330\overline{6}$ $\qquad f(0.2) = 0.182321556794$

$P_5(0.5) = 0.407291\overline{6}$ $\qquad f(0.5) = 0.405465108108$