Root Test

The final test for convergence or divergence is the **Root Test.** The Root Test works especially well for series that **involve n^{th} powers** and determines whether the series converges absolutely.

Root Test

Let $\sum a_n$ be a series.

1. $\sum a_n$ **converges absolutely** if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$
2. $\sum a_n$ **diverges** if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$
3. The Root Test is **inconclusive** if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Examples: Determine if the following series converge or diverge.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{e^2}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{e^2}{n}\right)\right|^n}$$

$$\lim_{n\to\infty} \frac{e^2}{n} = \frac{e^2}{\infty} = 0 < 1$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ converges absolutely

b. $\sum_{n=1}^{\infty} \frac{(-n)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{n^n}{(n+1)^{n+1}}$

$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(n+1)^{n+1}}}$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}}$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^n} \cdot \sqrt[n]{\frac{1}{n+1}}$

$\lim_{n\to\infty} \frac{n}{n+1} \cdot (n+1)^{-\frac{1}{n}} = 1$

$\ln y = \lim_{n\to\infty} -\frac{1}{n}\ln(n+1)$

$= \lim_{n\to\infty} \frac{-\ln(n+1)}{n}$

L'Hops $= \lim_{n\to\infty} \frac{-\frac{1}{n+1}}{1}$

$\ln y = 0$

$y = e^0 = 1$

$\therefore$ Root Test Inconclusive

$\sum \left|\frac{(-n)^n}{(n+1)^{n+1}}\right| = \sum \frac{n^n}{(n+1)^{n+1}}$ $b_n = \frac{n^n}{n^{n+1}} = \frac{1}{n}$

Divergent Harmonic

$\lim_{n\to\infty} \frac{\cancel{n^n}}{(n+1)^{n+1}} \cdot \frac{n^{n+1}}{\cancel{n^n}}$

$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1}$

$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \left(\frac{n}{n+1}\right) = \frac{1}{e}$

$\ln y = \lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right)$

$= \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{n^{-1}}$

L'Hops $= \lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)\left(\frac{1}{(n+1)^2}\right)}{-\frac{1}{n^2}}$

$= \lim_{n\to\infty} \frac{-n^2}{n(n+1)}$

$\ln y = -1$

$y = e^{-1} = \frac{1}{e}$

$\therefore \sum \frac{n^n}{(n+1)^{n+1}}$ Diverges by Limit Comparison

$\sum (-1)^n \frac{n^n}{(n+1)^{n+1}}$

$\lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}}$

$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \frac{1}{e} \cdot \frac{1}{\infty} = 0$

$\frac{(n+1)^{n+1}}{(n+2)^{n+2}} \le \frac{n^n}{(n+1)^{n+1}}$

$\frac{(n+1)^n(n+1)}{(n+2)^n(n+2)^2} \le \frac{n^n}{(n+1)^n(n+1)}$

$\left(\frac{n+1}{n+2}\right)^2 \le \left(\frac{n(n+2)}{(n+1)^2}\right)^n$

True

$\therefore \sum_{n=1}^{\infty} \frac{(-n)^n}{(n+1)^{n+1}}$ is conditionally convergent

c. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \frac{1}{\infty} = 0 < 1$$

$$\therefore \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n} \text{ converges Absolutely}$$

$$y = \lim_{n\to\infty} \sqrt[n]{n}$$

$$\ln y = \lim_{n\to\infty} \ln n^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \ln n$$

$$= \lim_{n\to\infty} \frac{\ln n}{n}$$

L'Hops $= \displaystyle\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{1}{\infty}$

$$\ln y = 0$$

$$y = e^0 = 1$$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
*n*th-Term	$\sum a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show Convergence
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$0<\|r\|<1$	$\|r\|>1$	Sum: $S = \frac{a_0}{1-r}$
Telescoping Series	$\sum a_n - a_{n+1}$	$\lim_{n\to\infty} S_n = S$	$\lim_{n\to\infty} S_n = \infty$	
p-Series	$\sum \frac{1}{n^p}$	$p>1$	$0<p<1$	

	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Alternating Series	$\sum(-1)^n a_n$	$\lim_{n\to\infty} a_n = 0$ $a_{n+1} \le a_n$		Remainder: $\lvert R_n \rvert \le a_{n+1}$
Integral (*f* is continuous, positive, and decreasing)	$\sum a_n$ $a_n = f(n) > 0$	$\int_1^\infty f(x)\,dx$ converges	$\int_1^\infty f(x)\,dx$ diverges	
Root	$\sum a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$ OR $= \infty$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1$
Ratio	$\sum a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$ OR $= \infty$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$

	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Direct Comparison **(a_n, $b_n > 0$)**	$\sum a_n$ $a_n \le b_n$	If $\sum b_n$ converges, then $\sum a_n$ converges	If $\sum a_n$ diverges, then $\sum b_n$ diverges	
Limit Comparison **(a_n, $b_n > 0$)**	$\sum a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} =$ FINITE & POSITIVE and b_n converges	$\lim_{n\to\infty} \frac{a_n}{b_n} =$ FINITE & POSITIVE and b_n diverges	