

Root Test

The final test for convergence or divergence is the **Root Test**. The Root Test works especially well for series that **involve n^{th} powers** and determines whether the series converges absolutely.

Root Test

Let $\sum a_n$ be a series.

1. $\sum a_n$ **converges absolutely** if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$
2. $\sum a_n$ **diverges** if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$
3. The Root Test is **inconclusive** if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$

Examples: Determine if the following series converge or diverge.

a.
$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{e^2}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{e^2}{n}\right)^n\right|}$$

$$\lim_{n \rightarrow \infty} \frac{e^2}{n} = \frac{e^2}{\infty} = 0 < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} \text{ converges absolutely}$$

$$b. \sum_{n=1}^{\infty} \frac{(-n)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{n^n}{(n+1)^{n+1}}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(n+1)^{n+1}}}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^n} \cdot \sqrt[n]{\frac{1}{n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot (n+1)^{-\frac{1}{n}} = 1$$

$$\begin{aligned} \ln y &= \lim_{n \rightarrow \infty} -\frac{1}{n} \ln(n+1) \\ &= \lim_{n \rightarrow \infty} -\frac{\ln(n+1)}{n} \\ &\stackrel{2^{\text{Hops}}}{=} \lim_{n \rightarrow \infty} \frac{-\frac{1}{n+1}}{1} \\ \ln y &= 0 \\ y &= e^0 = 1 \end{aligned}$$

∴ Root Test
Inconclusive

$$\sum \left| \frac{(-n)^n}{(n+1)^{n+1}} \right| = \sum \frac{n^n}{(n+1)^{n+1}} \quad b_n = \frac{n^n}{(n+1)^{n+1}} = \frac{1}{n}$$

Divergent
Harmonic

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^{n+1}} \cdot \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^{n+1}$$

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n \left(\frac{n}{n+1}\right) = \frac{1}{e}$$

∴ $\sum \frac{n^n}{(n+1)^{n+1}}$ Diverges
by Limit Comparison

$$\ln y = \lim_{n \rightarrow \infty} n \ln\left(\frac{n}{n+1}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$$

$$\stackrel{2^{\text{Hops}}}{=} \lim_{n \rightarrow \infty} \frac{\left(\frac{n+1}{n}\right) \left(\frac{1}{n+1}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{-\frac{1}{n^2}}{\frac{1}{n(n+1)}}$$

$$\ln y = -1 \\ y = e^{-1} = \frac{1}{e}$$

$$\sum (-1)^n \frac{n^n}{(n+1)^{n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^{n+1}}$$

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \frac{1}{e} \cdot \frac{1}{\infty} = 0$$

$$\frac{(n+1)^{n+1}}{(n+2)^{n+2}} \leq \frac{n^n}{(n+1)^{n+1}}$$

$$\frac{(n+1)^n (n+1)}{(n+2)^n (n+2)^2} \leq \frac{n^n}{(n+1)^n (n+1)}$$

$$\left(\frac{n+1}{n+2}\right)^2 \leq \left(\frac{n(n+2)}{(n+1)^2}\right)^n$$

True

∴ $\sum_{n=1}^{\infty} \frac{(-n)^n}{(n+1)^{n+1}}$ is conditionally
convergent

c.
$$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{(\ln n)^n}}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{\ln n} = \frac{1}{\infty} = 0 < 1$$

$$\therefore \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n} \text{ converges Absolutely}$$

$$y = \lim_{n \rightarrow \infty} \sqrt[n]{n}$$

$$\ln y = \lim_{n \rightarrow \infty} \ln n^{1/n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \ln n$$

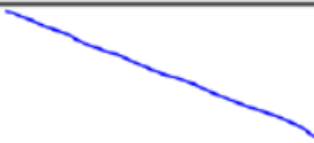
$$= \lim_{n \rightarrow \infty} \frac{\ln n}{n}$$

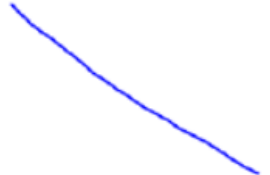
$$\begin{aligned} \text{L'Hop's} \\ = \lim_{n \rightarrow \infty} \frac{1/n}{1} = \frac{1}{\infty} \end{aligned}$$

$$\ln y = 0$$

$$y = e^0 = 1$$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
<i>n</i>th-Term	$\sum a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show <i>Convergence</i>
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$0 < r < 1$	$ r > 1$	Sum: $S = \frac{a_0}{1-r}$
Telescoping Series	$\sum a_n - a_{n+1}$	$\lim_{n \rightarrow \infty} S_n = S$	$\lim_{n \rightarrow \infty} S_n = \infty$	
<i>p</i>-Series	$\sum \frac{1}{n^p}$	$p > 1$	$0 < p < 1$	

	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Alternating Series	$\sum (-1)^n a_n$	$\lim_{n \rightarrow \infty} a_n = 0$ $a_{n+1} \leq a_n$		Remainder: $ R_n \leq a_{n+1}$
Integral (f is continuous, positive, and decreasing)	$\sum a_n$ $a_n = f(n) > 0$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	
Root	$\sum a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } > 1$ OR $= \infty$	Test is inconclusive if $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = 1$
Ratio	$\sum a_n$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1$ OR $= \infty$	Test is inconclusive if $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1$

	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Direct Comparison ($a_n, b_n > 0$)	$\sum a_n$ $a_n \leq b_n$	If $\sum b_n$ converges, then $\sum a_n$ converges	If $\sum a_n$ diverges, then $\sum b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum a_n$	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \text{FINITE}$ * POSITIVE and b_n converges	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \text{FINITE}$ * POSITIVE and b_n diverges	