Ratio Test

We previously discussed the fact that if the **absolute value** of a series **converges**, the series **converges.**

The **Ratio Test** is a test for **absolute convergence.**

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ **converges absolutely** if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$
2. $\sum a_n$ **diverges** if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$ or $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \infty$
3. The Ratio Test is **inconclusive** if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$

Examples: Determine the convergence or divergence of the following series.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2}{(n+1) \cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2}{n+1}$$

$$= 0 < 1$$

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ is absolutely convergent

b. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$$\lim_{n\to\infty} \left| \frac{(n+1)^2 \cdot 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2 \cdot 2^{n+1}} \right|$$

$$\lim_{n\to\infty} \frac{(n+1)^2 \cdot \cancel{2^n} \cdot \cancel{4}^{\,2}}{\cancel{3^n} \cdot 3} \cdot \frac{\cancel{3^n}}{n^2 \cdot \cancel{2^n} \cdot \cancel{2}}$$

$$\lim_{n\to\infty} \frac{2(n+1)^2}{3n^2}$$

L'Hops $\lim_{n\to\infty} \frac{4(n+1)}{6n}$

L'Hops $\lim_{n\to\infty} \frac{4}{6} = \frac{2}{3} < 1$

$\therefore \sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ converges absolutely by Ratio Test

c. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

1^{∞}

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$\lim_{n\to\infty} \frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{(n+1)} \cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e > 1$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ Diverges by Ratio Test

d. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty} \left| (-1)^{n+1} \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n \sqrt{n}} \right|$$

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2}$$

$$\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$$\sqrt{1+0} \cdot \frac{1+0}{1+0} = 1$$

Ratio Test Inconclusive

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} \qquad b_n = \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}} \rightarrow \text{Divergent p-series}$$

$$\lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{n+1} \cdot \frac{n}{\cancel{\sqrt{n}}}$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \rightarrow \text{Finite positive}$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ Diverges by Limit Comparison

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$

L'Hops

① $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$

② $\displaystyle\frac{d}{dn}\left[\frac{n^{1/2}}{n+1}\right] = \frac{(n+1)\frac{1}{2}n^{-1/2} - n^{1/2}}{(n+1)^2} = \frac{\frac{1}{2}n^{-1/2}[n+1-2n]}{(n+1)^2}$

$$= \frac{1-n}{2\sqrt{n}(n+1)^2} < 0$$

$$1-n<0$$
$$1<n$$

$\therefore \displaystyle\sum (-1)^n \frac{\sqrt{n}}{n+1}$ converges by Alt Series

$\therefore \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ is conditionally convergent

Why is the case $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ inconclusive? Consider the two familiar series:

Divergent Harmonic $\sum_{n=1}^{\infty}\frac{1}{n}$ $\qquad \lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\frac{n}{1}\right| = 1$

Convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $\qquad \lim_{n\to\infty}\left|\frac{1}{(n+1)^2}\cdot\frac{n^2}{1}\right| = 1$