2017 BC Question 5

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

(a) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2}$

$f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$

2 : $f'(3)$

(b) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2} = 0 \Rightarrow x = \dfrac{7}{4}$

The only critical point in the interval $1 < x < 2.5$ has x-coordinate $\frac{7}{4}$.

f' changes sign from positive to negative at $x = \frac{7}{4}$.

Therefore, f has a relative maximum at $x = \frac{7}{4}$.

2 : $\begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \quad\ \text{with justification} \end{cases}$

(c) $\displaystyle\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right)dx$

$\displaystyle = \lim_{b\to\infty}\Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty}\left[\ln\left(\frac{2x-5}{x-1}\right)\right]_5^b$

$\displaystyle = \lim_{b\to\infty}\left[\ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)\right] = \ln 2 - \ln\left(\frac{5}{4}\right) = \boxed{\ln\left(\frac{8}{5}\right)}$

3 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\displaystyle\int_5^\infty \frac{3}{2x^2-7x+5}\,dx$ converges.

— OR —

$\dfrac{3}{2n^2-7n+5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \ge 5$.

Since $\displaystyle\lim_{n\to\infty}\frac{\frac{3}{2n^2-7n+5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\displaystyle\sum_{n=5}^\infty \frac{1}{n^2}$ converges,

the series $\displaystyle\sum_{n=5}^\infty \frac{3}{2n^2-7n+5}$ converges by the limit comparison test.

2 : answer with conditions

Absolute and Conditional Convergence

Sometimes, a series with have both positive and negative terms and not be alternating. For instance:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \underbrace{\frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9}}_{\text{positive}} + \underbrace{\frac{\sin 4}{16} + \frac{\sin 5}{25} + \frac{\sin 6}{36}}_{\text{negative}} \cdots$$

has both positive and negative terms but is not alternating.

One way to investigate the convergence of this series is to look at the series:

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow \text{Convergent P-series}$$

$$0 \leq |\sin n| \leq 1$$

$$\therefore \sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| \text{ converges by direct comparison}$$

Absolute Convergence

If the series $\sum |a_n|$ **converges**, then the series $\sum a_n$ **also converges**.

The converse of this is not true. Consider the **alternating harmonic series.**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

~~If $\sum |a_n|$ diverges, $\sum a_n$ diverges~~

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$ Alternating Harmonic $\Longrightarrow$ Converges by Alternating Series Test

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$ Harmonic $\Longrightarrow$ Diverges by Integral Test

Absolute and Condition Convergence

1. $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ **converges.**

2. $\sum a_n$ is **conditionally convergent** if $\sum a_n$ **converges** but $\sum |a_n|$ **diverges.**

Example 1: Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n!}{2^n} \right| = \sum_{n=1}^{\infty} \frac{n!}{2^n}$$ Diverges by nth term test

$$\lim_{n\to\infty} \frac{n!}{2^n} = \infty \neq 0$$

* n! grows faster than 2^n

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$$

1) $\displaystyle\lim_{n\to\infty} \frac{n!}{2^n} = \infty \neq 0$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$ diverges by n^{th} term test

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightarrow \text{Divergent P-series}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

1) $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

2) $a_{n+1} \le a_n \rightarrow a_n$ are decreasing

$$\frac{d}{dn}\left[n^{-1/2}\right] = -\frac{1}{2}n^{-3/2} = \frac{-1}{n^{3/2}}$$

Always negative

$\therefore \sum \frac{(-1)^n}{\sqrt{n}}$ converges by Alternating Series test

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n(n+1)/2}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

$\downarrow$

Convergent Geometric

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n}$ is Absolutely convergent

d. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\ln(n+1)} \right| = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

$$\frac{1}{\ln(n+1)} \geq \frac{1}{n}$$

$$n \geq \ln(n+1)$$

$$e^n \geq n+1$$

$\sum \frac{1}{n}$ Divergent Harmonic

$\therefore \sum \frac{1}{\ln(n+1)}$ Diverges by Direct Comparison

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

1) $\displaystyle\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

2) $\displaystyle\frac{1}{\ln(n+2)} \leq \frac{1}{\ln(n+1)}$

$$\ln(n+1) \leq \ln(n+2)$$

$$n+1 \leq n+2$$

$$1 \leq 2$$

$\therefore \sum \frac{(-1)^n}{\ln(n+1)}$ converges by Alt Series Test

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ is conditionally Convergent

e. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \qquad b_n = \frac{1}{2\sqrt{n}}$$

Divergent p-series

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{2\sqrt{n}}{1}$$

$$\lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n}}+1} = 1$$

$\therefore \sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$ Diverges by Limit Comparison

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$$

1) $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

2) $\displaystyle\frac{1}{\sqrt{n+2}+\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$$\cancel{\sqrt{n+1}} + \sqrt{n} \leq \sqrt{n+2} + \cancel{\sqrt{n+1}}$$

$$n \leq n+2$$

$$0 \leq 2$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$ Converges Conditionally

Rearrangement of Series

A finite series can be rearranged without changing the value of the sum. This is not necessarily true of an infinite series.

- If the series is **absolutely convergent, every rearrangement has the same sum**
- If the series is **conditionally convergent, the sum depends on the arrangement of the terms.**

Example 2: Consider the following series:

a. $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \boxed{\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\dots\right)} = \ln 2$$

$$= \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)\dots$$

$$= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}\dots$$

$$= \frac{1}{2}\boxed{\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\dots\right)}$$

$$= \frac{1}{2}\ln 2$$