Alternating Series

For most of the tests we have used, the series have involved **positive terms.** Some series have terms that are both positive and negative. The simplest of such series an **alternating series,** whose terms alternate signs

$\sum a r^n$

Example: $$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} = \sum_{n=0}^{\infty}-\frac{1}{2}\left(-\frac{1}{2}\right)^n$$

Convergent geometric $|r|<1$

$$S = \frac{-1/2}{1- -1/2} = -\frac{1}{3}$$

- Alternating series occur when either the odd terms are negative or the even terms are negative.

Alternating Series Test

Let $\boxed{a_n > 0}$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim\limits_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ *for all n*

If FALSE, Alt series test does not apply.

∴ Divergent by n^{th} term test.

At some point

Example 1: Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}}_{\text{Alternating Harmonic}}$

① $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$

② $a_{n+1} \le a_n$

$\dfrac{1}{n+1} \le \dfrac{1}{n}$

$n \le n+1$

$\boxed{0 \le 1}$

$\downarrow$ Always True

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by Alt. Series test.

Example 2: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$

$$= \sum_{n=1}^{\infty} \frac{n}{(-1\cdot 2)^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{n}{(-1)^{n-1}\cdot 2^{n-1}}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2n}{2^n}$$

① $\lim_{n\to\infty} \frac{2n}{2^n}$

L'Hops $\lim_{n\to\infty} \frac{2}{\ln 2 \cdot 2^n} = \frac{2}{\infty} = 0$

② $\frac{2(n+1)}{2^{n+1}} \leq \frac{2n}{2^n}$

$\cancel{2^{n+1}}(n+1) \leq 2n \cdot \cancel{2^{n+1}}$

$n+1 \leq 2n$

$1 \leq n$

$\therefore \sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$ Converges by Alt. Series Test

Example 3: Cases in which the Alternating Series Test Fails

$\frac{1}{2} < \frac{2}{3}$ $\frac{1}{3} < \frac{2}{4} \Rightarrow a_{n+1} > a_n$

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$

① $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

Alt Series Fails

$\therefore$ Diverges by n^{th} term test

b. $\left(\frac{2}{1} - \frac{1}{1}\right) + \left(\frac{2}{2} - \frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \left(\frac{2}{4} - \frac{1}{4}\right), \ldots$

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{n} - \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

Divergent Harmonic

Alternating Series Remainder

For **convergent alternating series**, the partial sum S_n can be a useful approximation for the sum S of the series.

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_n involved in approximating the sum of the series by a partial sum is less than (or equal to) the first neglected term. That is,

$$\underbrace{S}_{\text{Actual Sum of } \sum(-1)^n a_n} - \underbrace{S_n}_{n^{th} \text{ partial sum}} \quad |R_n| \le \boxed{a_{n+1}}$$

$$|S - S_n| = \underbrace{|R_n|}_{\text{Remainder (error)}} \le \underbrace{\boxed{a_{n+1}}}_{\text{First Omitted term}}$$

Actual Sum of $\sum(-1)^n a_n$

n^{th} partial sum

Remainder (error)

First Omitted term

Example 4: Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

① $\lim_{n\to\infty} \frac{1}{2n-1} = 0$

② $\frac{1}{2(n+1)-1} \leq \frac{1}{2n-1}$

$2n-1 \leq 2n+1$

$\boxed{-1 \leq 1}$

Always True

$S \approx S_6 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$

$= \frac{2578}{3465} = 0.744012$

$|S - S_6| = |R_6| \leq \overset{a_7}{\frac{1}{13}} = 0.076923$

$0.744012 - 0.076923 \leq S \leq 0.744012 + 0.076923$

$0.667089 \leq S \leq 0.820935$

$0.667 \leq S \leq 0.821$

Example 5: Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n!}\right)$$

① $\lim_{n\to\infty} \frac{1}{n!} = 0$

② $\frac{1}{(n+1)!} \le \frac{1}{n!}$

$n! \le (n+1)!$

$\cancel{n!} \le (n+1)\cdot \cancel{n!}$

$1 \le n+1$

$0 \le n$

$S_6 = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}$

$= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720}$

$= 0.631944$

$|S - 0.631944| \le \frac{1}{7!} = \frac{1}{5040} = 0.000198$

$0.631944 - 0.000198 \le S \le 0.631944 + 0.000198$

$0.631746 \le S \le 0.632142$

Example 6: Determine the number of terms required to approximate the sum of the convergent series with an error less than 0.001.

$$\sum_{\boxed{n=0}}^{\infty} \frac{(-1)^n}{(2n+1)!} = \sin(1) = 0.8415$$

$$|S - S_n| = |R_n| \le \boxed{a_{n+1} < \frac{1}{1000}}$$

$$\frac{1}{(2(n+1)+1)!} < \frac{1}{1000}$$

$$\frac{1}{(2n+3)!} < \frac{1}{1000}$$

$$1000 < (2n+3)!$$

$$2n+3 = 7$$

$$\boxed{n = 2}$$

3 TERMS ARE NECESSARY.

$$S_2 = 1 - \frac{1}{3!} + \frac{1}{5!} = 0.841\overline{6}$$

$$|\sin(1) - S_2| = 0.0001957$$