For the series we have looked at so far, to determine convergence or divergence, they had to have special characteristics. A slight change in the series could make the tests not applicable.

For Example: Consider the following series.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n}$ a geometric series. $\displaystyle\sum_{n=0}^{\infty}\frac{n}{2^n}$ not a geometric series

2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ a p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3+n}$ not a p-series

3. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n^2+3)^2}$ easy application of integral test $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{(n^2+3)^2}$ not easily integrable

Because of the behavior of convergent and divergent series, we can compare a complicated series with a simpler series to determine convergence or divergence

Direct Comparison Test

positive Series

Let $\boxed{0 < a_n \leq b_n}$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ **converges**, then $\sum_{n=1}^{\infty} a_n$ **converges.**
2. If $\sum_{n=1}^{\infty} a_n$ **diverges**, then $\sum_{n=1}^{\infty} b_n$ **diverges.**

Example 1: Using a direct comparison, determine whether the following series converge or diverge.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

If $2+3^n > 3^n$, then

$$\frac{1}{2+3^n} < \frac{1}{3^n}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ convergent geometric

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ convergent by a direct comparison

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

$2+\sqrt{n} < n$ for $n>4$

$2+\sqrt{n} = n$

$0 = n^1 - n^{1/2} - 2$

$0 = (n^{1/2}-2)(n^{1/2}+1)$

$n^{1/2}-2=0$ $\quad \cancel{n^{1/2}+1=0}$

$n=4$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}} > \boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$ for $n>4$ → Divergent Harmonic

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ Diverges by Direct Comparison

c. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

$$\frac{1}{e^{n^2}} < \frac{1}{e^n}$$

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ convergent geometric

$\therefore \displaystyle\sum_{n=0}^{\infty} e^{-n^2}$ converges by direct comparison

Limit Comparison Test

Sometimes, establishing a term by term comparison is not easy or possible. When this happens we can apply a second comparison test.

Suppose that $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \boxed{L}$ where L is *finite and positive*.

Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 2: Show that the general harmonic series $\sum_{n=1}^{\infty} \frac{1}{an+b}$, where $a > 0$ *and* $b > 0$, diverges.

$a_n = \frac{1}{an+b}$ $\qquad b_n = \frac{1}{n} \longrightarrow$ Divergent Harmonic (Integral test)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{an+b}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{an+b} = \boxed{\frac{1}{a}}$$

$\rightarrow$ Finite & Positive

$\therefore \sum_{n=1}^{\infty} \frac{1}{an+b}$ Diverges by Limit Comparison

The Limit Comparison Test works well for comparing a "messy" algebraic series with a p-series. In choosing an appropriate p-series, choose one with an **n**$^{\text{th}}$ **term of the same magnitude.** You can **disregard all but the highest powers** of n in the numerator and denominator.

Example 3: Determine whether the following series converge or diverge.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$

$a_n = \frac{1}{\sqrt{3n-2}}$

$b_n = \frac{1}{\sqrt{n}}$ → DIVERGENT p-series ($p = 1/2$)

$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{3n-2}}$

$\lim_{n\to\infty} \sqrt{\frac{n}{3n-2}}$

$\sqrt{\lim_{n\to\infty} \frac{n}{3n-2}} = \boxed{\sqrt{1/3}}$

Finite & positive

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$ Diverges by Limit Comparison

b. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

$a_n = \frac{\sqrt{n}}{n^2+1}$ $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ → Convergent p-series ($p = 3/2$)

$\lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{n^2+1} \cdot \frac{n^2}{\cancel{\sqrt{n}}}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \boxed{1}$ Finite & Positive

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges by Limit comparison

Example 4: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$.

$a_n = \frac{n2^n}{4n^3+1}$

$b_n = \boxed{\frac{2^n}{n^3}}$

$\lim_{n\to\infty} \frac{2^n}{n^3} = \infty \neq 0$

$\therefore$ Diverges by nth term test

$\lim_{n\to\infty} \frac{n \cdot \cancel{2^n}}{4n^3+1} \cdot \frac{n^3}{\cancel{2^n}}$

$\lim_{n\to\infty} \frac{n^4}{4n^3+1} = \boxed{\infty}$

Not Finite

$\therefore$ Inconclusive

$a_n = \frac{n2^n}{4n^3+1}$

$b_n = \frac{n2^n}{n^3} = \boxed{\frac{2^n}{n^2}}$

$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$

$\lim_{n\to\infty} \frac{\cancel{n2^n}}{4n^3+1} \cdot \frac{n^3}{\cancel{n2^n}}$

$\lim_{n\to\infty} \frac{n^3}{4n^3+1} = \boxed{\frac{1}{4}}$ Finite & Positive

$\therefore \sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$ Diverges by Limit Comparison