Infinite Series: Continued

Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $\boxed{a_n} = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

Example 1: Determine whether the following sequence converge or diverge using the integral test.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

pos ✓
CONT ✓
DEC ✓

$$\frac{d}{dn}\left[\frac{n}{n^2+1}\right] = \frac{n^2+1-n(2n)}{(n^2+1)^2}$$

$$= \frac{-(n^2-1)}{(n^2+1)^2} < 0$$

For $n > 1$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\int \frac{1}{u}\,du$

$\frac{1}{2}\ln|u|$

$$\lim_{b\to\infty}\left[\frac{1}{2}\ln|x^2+1|\right]_1^b$$

$$\frac{1}{2}\lim_{b\to\infty}\left[\ln|b^2+1| - \ln(2)\right] = \infty$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ Diverges by the integral test

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

pos ✓
cont ✓
DEC ✓

$$\frac{d}{dn}\left[(n^2+1)^{-1}\right]$$

$$-(n^2+1)^{-2}\cdot 2n$$

$$\frac{-2n}{(n^2+1)^2} < 0$$

for $n > 0$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx$$

$$\lim_{b\to\infty}\left[\arctan x\right]_1^b$$

$$\lim_{b\to\infty}\left[\arctan b - \arctan 1\right]$$

$$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Converges by the integral test

c. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

cont ✓
pos ✓
DEC ✓

$$\int_2^{\infty} \frac{1}{\ln x} \cdot \frac{1}{x}\,dx$$

$u = \ln x$
$du = \frac{1}{x}dx$
$\int \frac{1}{u}\,du$
$\ln|u|$

$$\lim_{b\to\infty} \Big[\ln|\ln x|\Big]_2^b$$

$$\lim_{b\to\infty} \Big[\ln|\ln b| - \ln|\ln 2|\Big] = \infty$$

$\therefore \displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Diverges by integral test

d. $\displaystyle\sum_{n=3}^{\infty} n^2 e^n$

pos ✓
cont ✓
DEC ✗

Integral test does not apply

$$\cancel{\int_3^{\infty} x^2 e^x\,dx}$$

$$\lim_{n\to\infty} n^2 e^n = \infty \cdot \infty = \infty \neq 0$$

$\therefore \displaystyle\sum_{n=3}^{\infty} n^2 e^n$ Diverges by n^{th} term test

$p-Series$ and Harmonic Series

A $p-Series$ is a series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}+\cdots$

When $p=1$, the series $\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+\cdots$ is known as the harmonic series.

-We can show the harmonic series is divergent using the integral test.

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$

$$\lim_{b\to\infty}\Big[\ln|x|\Big]_1^b=\infty$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n} \text{ Diverges}$$

General Harmonic

$$\sum_{n=1}^{\infty}\frac{1}{An+B}\ ,\quad A \text{ \& } B \text{ are constants}$$

$$\lim_{b\to\infty}\int_1^b \frac{1}{Ax+B}\,dx$$

$$\lim_{b\to\infty}\left[\frac{1}{A}\ln|Ax+B|\right]_1^b=\infty$$

-We can discuss the convergence and divergence of p-series using the integral test as well.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\frac{b^1}{b^{1/2}} = b^{1/2} =$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx$$

$$\lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$\lim_{b\to\infty} \left[\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$$

$$\frac{1}{-p+1} \lim_{b\to\infty} \left[\frac{b^1}{b^p} - 1\right]$$

$\to p > 1$ CONVERGENT INTEGRAL

$\to p < 1$ Divergent Integral

$p < 1$

$\frac{1}{1}$ $p > 1$

Convergence of $p - Series$

Any series of the form of a $p - Series$: 1) Converges if $p > 1$

2) Diverges if $0 < p < 1$

Example 3: Determine if the following sequences converge or diverge.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

$p = 4 > 1$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ convergent p-series

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

$p = 1/5 < 1$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ Divergent p-series

c. $\overset{a_1}{1} + \overset{a_2}{\left(1 - \frac{3}{4}\right)} + \overset{a_3}{\left(1 - \frac{8}{9}\right)} + \overset{a_4}{\left(1 - \frac{15}{16}\right)} + \cdots$

$\displaystyle\sum_{n=1}^{\infty} 1 - \frac{(n^2 - 1)}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} 1 - \frac{n^2}{n^2} + \frac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$p = 2 > 1$

$\therefore$ Convergent p-series