Series

Infinite sequences are necessary to study infinite summations, i.e. **infinite series.**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Many times, to consider the sum of an infinite series, we can focus on the **sequence of partial sums.**

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

Definition of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the **nth partial sum** is given by $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

- **If the sequence of partial sums $\{S_n\}$ converges to S**, then the series $\sum_{n=1}^{\infty} a_n$ **converges**. The **limit S** is called the **sum of the series.**

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- **If $\{S_n\}$ diverges**, then the series **diverges.**

Example 1: Determine if the following series converges.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$

$$S_1 = \boxed{\frac{1}{2}}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \boxed{\frac{7}{8}}$$

$$S_4 = \frac{7}{8} + \frac{1}{16} = \boxed{\frac{15}{16}}$$

$$\vdots$$

$$S_n = \frac{2^n - 1}{2^n}$$

L'Hops

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} \quad \frac{\infty}{\infty} \quad = \lim_{n\to\infty} \frac{\ln 2 \cdot 2^n}{\ln 2 \cdot 2^n} = 1$$

$$= \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$$

Example 2: Determine if the following **telescoping series** converges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1$$

$S_1 = 1 - \frac{1}{2} = \boxed{\frac{1}{2}} = 1 - \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \boxed{\frac{2}{3}} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = 1 - \frac{1}{3}$

$S_3 = \frac{2}{3} + \frac{1}{3} - \frac{1}{4} = \boxed{\frac{3}{4}} = 1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = 1 - \frac{1}{4}$

$S_4 = \frac{3}{4} + \frac{1}{4} - \frac{1}{5} = \boxed{\frac{4}{5}} = 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = 1 - \frac{1}{5}$

$\vdots \qquad \vdots$

$S_n = \frac{n}{n+1}$ $\qquad S_n = 1 - \frac{1}{n+1}$

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$ $\qquad \lim\limits_{n\to\infty} 1 - \frac{1}{n+1} = 1$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$S_1 = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$ $\qquad = 1 - \frac{1}{3}$

$S_2 = \frac{2}{3} + \frac{1}{2} - \frac{1}{4} = \boxed{\frac{11}{12}}$ $\qquad = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$

$S_3 = \frac{11}{12} + \frac{1}{3} - \frac{1}{5} = \boxed{\frac{21}{20}} = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$

$S_4 = \frac{21}{20} + \frac{1}{4} - \frac{1}{6} = \boxed{\frac{17}{15}} = 1 + \frac{1}{2} - \cancel{\frac{1}{4}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6} = 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$

$S_5 = \frac{17}{15} + \frac{1}{5} - \frac{1}{7} = \boxed{\frac{25}{21}} = 1 + \frac{1}{2} - \cancel{\frac{1}{5}} - \frac{1}{6} + \cancel{\frac{1}{5}} - \frac{1}{7} = 1 + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}$

$S_6 = \frac{25}{21} + \frac{1}{6} - \frac{1}{8} = \boxed{\frac{69}{56}}$

$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$

$\lim\limits_{n\to\infty} S_n = 1 + \frac{1}{2} - 0 - 0$

$= \boxed{\frac{3}{2}}$

Example 3: A telescoping series disguised.

$$\sum_{n=1}^{\infty}\left(\frac{4}{4n^2-1}\right) = \sum_{n=1}^{\infty} \frac{2}{2n-1} - \frac{2}{2n+1}$$

$$\frac{4}{(2n+1)(2n-1)} = \frac{A}{2n+1} + \frac{B}{2n-1}$$

$$4 = (2A+2B)n + (-A+B)$$

$$2A+2B=0$$

$$2(-A+B=4)$$

$$4B=8$$

$$\boxed{B=2} \quad \boxed{A=-2}$$

$$S_1 = 2 - \frac{2}{3}$$

$$S_2 = 2 - \frac{2}{3} + \frac{2}{3} - \frac{2}{5} = 2 - \frac{2}{5}$$

$$S_3 = 2 - \frac{2}{5} + \frac{2}{5} - \frac{2}{7} = 2 - \frac{2}{7}$$

$$\vdots$$

$$S_n = 2 - \frac{2}{2n+1}$$

$$\lim_{n\to\infty} S_n = 2 - 0 = \boxed{2}$$

Geometric Series

One very common type of series is a **Geometric Series.** In general, the series is given by

- We say this is geometric series with **ratio r.**

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a \neq 0$$

Convergence of Geometric Series

A geometric series with ratio r:

1) Diverges if $|r| \geq 1$.

2) Converges if $0 < |r| < 1$. The series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} ar^n$$

Example 4: Determine if the following geometric series' converge. If the series converges, find the sum.

a. $$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3 \cdot \frac{1^n}{2^n}$$

$$= \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$

CONVERGENT GEOMETRIC

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \frac{3}{1-\frac{1}{2}} = 6$$

b. $$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n = \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{3}{2}\right)^n$$

Divergent

c. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n} = \sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{4}\right)^n$

$\downarrow$ Convergent

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

d. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{4^n} = \frac{4}{3} - (a_0 + a_1 + a_2)$

$$= \frac{4}{3} - \left(1 + \frac{1}{4} + \frac{1}{16}\right)$$

$$= \frac{4}{3} - \frac{21}{16}$$

$$= \frac{1}{48}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n+3}$$

$$\sum_{n=0}^{\infty} \frac{1}{64}\left(\frac{1}{4}\right)^n = \frac{\frac{1}{64}}{1-\frac{1}{4}}$$

$$= \frac{1}{\cancel{64}_{16}} \cdot \frac{\cancel{4}^{1}}{3}$$

$$= \frac{1}{48}$$

e. $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{5^n} = \sum_{n=2}^{\infty} \left(\frac{2}{5}\right)^n = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^{n+2}$

$\displaystyle = \sum_{n=0}^{\infty} \frac{4}{25}\left(\frac{2}{5}\right)^n$

Convergent

$$\sum_{n=2}^{\infty} \frac{2^n}{5^n} = \frac{\frac{4}{25}}{1-\frac{2}{5}}$$

$$= \frac{4}{\cancel{25}\,5} \cdot \frac{\cancel{5}\,1}{3}$$

$$= \frac{4}{15}$$

f. $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} \frac{9^2}{9^n} \cdot 4^n \cdot 4$

$\displaystyle = \sum_{n=1}^{\infty} 324\left(\frac{4}{9}\right)^n$

$\displaystyle = \sum_{n=0}^{\infty} 324\left(\frac{4}{9}\right)^{n+1}$

$\displaystyle = \sum_{n=0}^{\infty} 144\left(\frac{4}{9}\right)^n$

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \frac{144}{1-\frac{4}{9}}$$

$$= 144 \cdot \frac{9}{5}$$

$$= \frac{1296}{5}$$

g. $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=1}^{\infty} \frac{\left((-4)^3\right)^n}{\frac{5^n}{5}} = \sum_{n=1}^{\infty} 5\left(-\frac{64}{5}\right)^n$

Diverges

nth – Term Test for $\boxed{\text{Divergence}}$ (ALWAYS CHECK FIRST)

There are two statements that we can conclude about a series based upon the limit of the nth term.

1) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Example: $\sum_{n=1}^{\infty} \frac{1}{2^n}$ → CONVERGENT GEOMETRIC SINCE $|r| = \frac{1}{2} < 1$

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

2) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \qquad \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

Example: $\sum_{n=1}^{\infty} 2^n$ $\qquad \lim_{n\to\infty} 2^n = \infty \neq 0$

- No other form of the statement is necessarily true.

Example: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$

$$* \lim_{n\to\infty} \frac{1}{n} = 0$$

nth Term test is inconclusive

Example 1: Determine whether the following sequences diverge.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 - n}{n^2 - 9}$

$\displaystyle\lim_{n\to\infty} \frac{2n^2-n}{n^2-9} = 2 \neq 0$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{2n^2-n}{n^2-9}$ Diverges by n^{th} term test

b. $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$

$\displaystyle\lim_{k\to\infty} \ln\left(\frac{k+1}{k}\right)$

$\ln\left[\displaystyle\lim_{k\to\infty} \frac{k+1}{k}\right]$

$\ln[1] = 0$

$\therefore$ n^{th} term test is inconclusive

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 4^n}{1 + 3^n}$

$\displaystyle\lim_{n\to\infty} \frac{1+4^n}{1+3^n} = \infty$

L'Hops

$\displaystyle\frac{\ln 4}{\ln 3} \cdot \lim_{n\to\infty} \left(\frac{4}{3}\right)^n$

$y = \displaystyle\lim_{n\to\infty} \left(\frac{4}{3}\right)^n$

$\ln y = \displaystyle\lim_{n\to\infty} n \ln\left(\frac{4}{3}\right)$

$\ln y = \infty \cdot \ln\left(\frac{4}{3}\right)$

$e^{\ln y} = e^{\infty}$

$y = e^{\infty} = \infty$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ Diverges by n^{th} term