Sequences

In mathematics, a **sequence** is defined as a function whose **domain is the set of positive integers.** Most of the time, the **range** of a sequence is a set of numbers that **follows a certain pattern** based upon the values of the domain.

Term # $\longrightarrow$ $1, \quad 2, \quad 3, \quad \ldots, \quad n, \quad n+1, \quad \ldots,$

$\downarrow \quad \downarrow \quad \downarrow \quad \quad \downarrow \quad \downarrow$

Term $\longrightarrow$ $a_1, \quad a_2, \quad a_3, \quad \ldots, \; a_{n-1}, \; a_n, \quad a_{n+1}, \quad \ldots,$

1st Term 2nd Term nth term (NEXT TERM)

Example 1:

a. The terms in a sequence can be defined explicitly, referencing the term number:

$$\{a_n\} = \left\{\frac{n^2}{2^n - 1}\right\}$$

$a_1 = \frac{1^2}{2^n - 1}$ $a_2 = \frac{2^2}{2^2 - 1}$ $a_3 = \frac{3^2}{2^3 - 1}$

$a_1 = 1$ $a_2 = \frac{4}{3}$ $a_3 = \frac{9}{7}$ $\ldots$

b. The terms can be defined recursively, referencing the previous terms in the sequence:

$$b_1 = 1, \quad b_2 = 1, \quad \{ b_{n+1}\} = \{b_{n-1} + b_n\}$$

Fibonnaci Sequence $\longrightarrow$ $b_3 = 1 + 1$ $b_4 = 1 + 2$ $b_5 = 2 + 3$ $b_6 = 3 + 5$

$b_3 = 2$ $b_4 = 3$ $b_5 = 5$ $b_6 = 8$

The primary focus of this part of calculus concerns sequences whose **terms approach limiting values**. If a sequence approaches some limiting value, the sequence is said to **converge.**

Definition of the Limit of a Sequence

Let L be a real number. The limit of a sequence $\{a_n\}$ is L can be written: $\lim_{n\to\infty}\{a_n\} = L$

Example 2: Find the limit of the sequence whose n^{th} term is

$\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}$ $\quad a_1 = 2 \quad a_2 = \frac{9}{4} \quad a_3 = \frac{64}{27} \quad a_4 = \frac{625}{256} \quad \ldots$

$$y = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \qquad 1^\infty$$

$$\ln y = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right) \qquad \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\ln(1+n^{-1})}{n^{-1}}$$

L'Hops

$$= \lim_{n\to\infty} \frac{\frac{1}{1+n^{-1}} \cdot \cancel{-n^{-2}}}{\cancel{-n^{-2}}}$$

$$\ln y = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$e^{\ln y} = e^1$$

$$y = e^1$$

Example 3: Determine if the following sequences converge or diverge.

a. $\{a_n\} = \{3 + (-1)^n\}$

If n is odd, $a_n = 3 - 1 = 2$

If n is even, $a_n = 3 + 1 = 4$

$\lim_{n\to\infty} a_n = \text{DNE}$

$\therefore a_n$ Diverges

b. $\{a_n\} = \left\{\frac{n^2}{2^n - 1}\right\}$

$\lim_{n\to\infty} \frac{n^2}{2^n - 1}$ $\quad \frac{\infty}{\infty}$

L'Hops $\lim_{n\to\infty} \frac{2n}{\ln 2 \cdot 2^n}$ $\quad \frac{\infty}{\infty}$

L'Hops $\lim_{n\to\infty} \frac{2}{(\ln 2)^2 \cdot 2^n} = \frac{2}{\infty} = 0$

$x = 10$

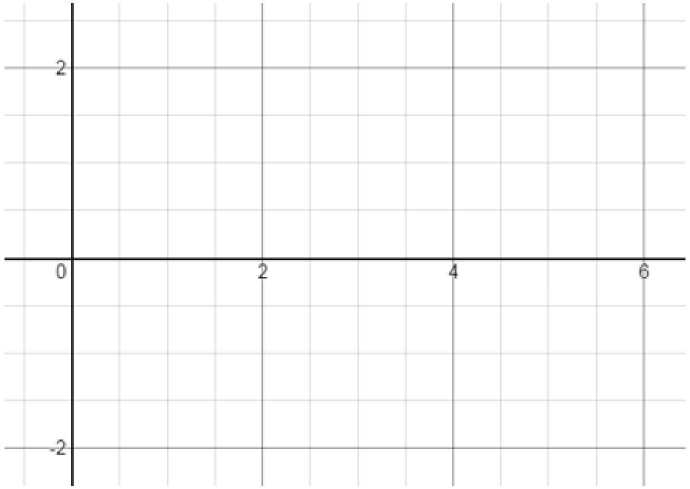
$2^x \quad x!$

$\uparrow \quad \uparrow$

FASTEST $\longrightarrow$ SLOWEST

GROWTH FACTORS: $x^x,\ x!,\ a^x,\ \underset{(a\geq 2)}{x^a},\ x\log x,\ \underset{(0<a<2)}{x^a},\ \log x,\ c$

Example 4: Sometimes, we need to use the **Squeeze Theorem** to determine the convergence of a sequence. Determine if the sequence $\{a_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$.



Example 5: In the previous example, we saw a sequence whose terms alternated between positive and negative values. When this happens, we can apply the **Absolute Value theorem** for sequences to help determine convergence.

For the sequence $\{a_n\}$, if $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$

$$\{a_n\} = \left\{(-1)^n \left(\frac{3^n - 1}{n!}\right)\right\}$$

$$|a_n| = \frac{3^n - 1}{n!}$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n - 1}{n!} = 0 \quad \frac{\infty}{\infty}$$

$$\lim_{n\to\infty} (-1)^n \left(\frac{3^n - 1}{n!}\right) = 0$$

A sequence is said to be **monotonic** if its terms are **non- decreasing or non-increasing.**

ALWAYS INCREASING   ALWAYS DECREASING

Example 6: Determine if the following sequences are monotonic.

a. $\{a_n\} = \{3 + (-1)^n\}$ Non - Monotonic

b. $\{b_n\} = \left\{\frac{2n}{1+n}\right\}$

$b_n < b_{n+1}$

$$\frac{2n}{1+n} < \frac{2(n+1)}{1+(n+1)} \Rightarrow \frac{2n}{1+n} < \frac{2n+2}{n+2}$$

$$2n^2+4n < 2n^2+4n+2$$

$$0 < 2$$

Always TRUE

Monotonic (INCREASING)

c. $\{c_n\} = \left\{\frac{n^2}{2^n-1}\right\}$

$C_1 = 1$

$C_2 = \frac{4}{3}$

$C_3 = \frac{9}{7}$

$C_4 = \frac{16}{15}$

$C_n > C_{n+1}$

$$\frac{n^2}{2^n-1} > \frac{(n+1)^2}{2^{n+1}-1}$$

$$n^2(2^{n+1}-1) > (n+1)^2(2^n-1)$$

$$\frac{d}{dn}\left[\frac{n^2}{2^n-1}\right] = \frac{\boxed{(2^n-1)\cdot 2n - n^2(\ln 2 \cdot 2^n)}}{(2^n-1)^2}$$

$$n\left[2\cdot 2^n - 2 - n\ln 2\cdot 2^n\right]$$

$$n\left[2\cdot 2^n - \ln 2^n \cdot 2^n - 2\right]$$

$$\frac{d}{dn}\left[\frac{n^2}{2^n-1}\right] < 0 \quad \text{for } n \geq 2$$

C_n is monotonic for $n \geq 2$

Definition of a bounded sequence

A sequence is **bounded** if there is some number such that the sequence is **always smaller and always larger** than that number.

- A sequence is bounded above if it is always smaller than some number. $\{a_n\} \leq M$ for all n

- A sequence is bounded below if it is always larger than some number. $\{a_n\} \geq N$ for all n

If a sequence is **bounded and monotonic, then it converges.**

Example 7: Determine if the following sequences converge or diverge.

a. $\{a_n\} = \left\{\frac{n}{n+1}\right\}$

b. $\{b_n\} = \left\{\frac{n^2}{n+1}\right\}$

Example 8: A sequence is defined by $a_1 = 0$ and $\boxed{a_{n+1} = \sqrt{a_n + 6}}$ for all $n \geq 1$. Show that this sequence converges by showing it is bounded and monotonic.

$a_2 = \sqrt{6}$ $a_3 = \sqrt{\sqrt{6} + 6}$ $a_4 = \sqrt{\sqrt{\sqrt{6}+6} + 6}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = B$

As $n \to \infty$,

$B^2 = \sqrt{B+6}^{\,2}$

$B^2 = B + 6$

$B^2 - B - 6 = 0$

$(B-3)(B+2) = 0$

$\boxed{B = 3}$ $\cancel{-2}$

BOUNDED

$a_n = \sqrt{a_n^2} = \sqrt{a_n \cdot a_n}$

$a_n < 3$

$a_n < \sqrt{3 \cdot a_n}$

$a_n < \sqrt{a_n + 2a_n}$

$2a_n < 6$

$a_n < \boxed{\sqrt{a_n + 6}}$

$a_n < a_{n+1}$

Monotonic

(increasing)

Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

1. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\right\}$

2. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\right\}$

3. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$

4. $\{-1, 2, 7, 14, 23, \ldots\}$