Integration by Trig Substitution

For integrals that **involve radicals**, sometimes none of our rules or applications will fit the given integral. When that is the case, we can use **right triangle trig** to help us find a trigonometric function to **substitute for the radical** in the integral.

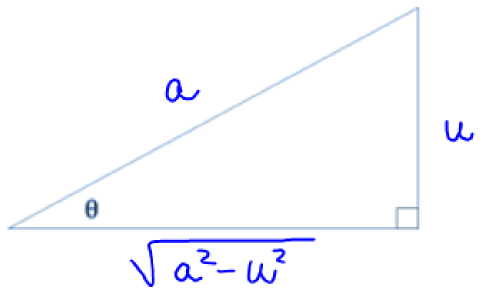
Trigonometric Substitutions (a>0)

u is a function of x

1. Integrals involving $\boxed{\sqrt{a^2 - u^2}}$:

$$\sin\theta = \frac{u}{a}$$

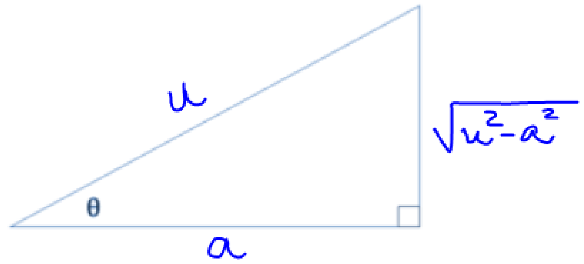
$$u = a\sin\theta$$



2. Integrals involving $\boxed{\sqrt{u^2 - a^2}}$:

$$\sec\theta = \frac{u}{a}$$

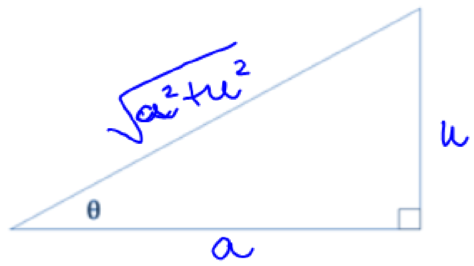
$$u = a\sec\theta$$



3. Integrals involving $\boxed{\sqrt{a^2 + u^2}}$

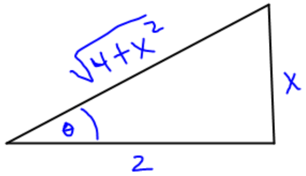
$$\tan\theta = \frac{u}{a}$$

$$u = a\tan\theta$$



Example 1:

$$\int \frac{dx}{\sqrt{4+x^2}}$$



$\tan\theta = \frac{x}{2}$

$x = 2\tan\theta \rightarrow dx = 2\sec^2\theta \, d\theta$

$$\int \frac{2\sec^2\theta}{\sqrt{4+(2\tan\theta)^2}} \, d\theta$$

$$\int \frac{2\sec^2\theta}{\sqrt{4(1+\tan^2\theta)}} \, d\theta$$

$$\int \frac{\cancel{2}\sec^2\theta}{\cancel{\sqrt{4}} \cdot \sqrt{1+\tan^2\theta}} \, d\theta$$

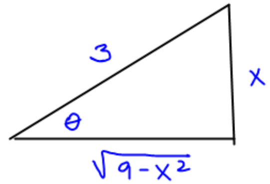
$$\int \sec\theta \, d\theta$$

$$\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C \quad \text{OR} \quad \ln\left|\sqrt{4+x^2} + x\right| + C}$$

Example 2:

$$\int \frac{dx}{x^2\sqrt{9-x^2}}$$



$\sin\theta = \frac{x}{3}$

$x = 3\sin\theta \rightarrow dx = 3\cos\theta \, d\theta$

$$\int \frac{3\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} \, d\theta$$

$$\int \frac{3\cos\theta}{9\sin^2\theta \cdot \sqrt{9} \cdot \sqrt{1-\sin^2\theta}} \, d\theta$$

$$\frac{1}{9}\int \frac{1}{\sin^2\theta} \, d\theta$$

$$\frac{1}{9}\int \csc^2\theta \, d\theta$$

$$-\frac{1}{9}\cot\theta + C$$

$$-\frac{1}{9} \cdot \frac{\sqrt{9-x^2}}{x} + C$$

$$\boxed{-\frac{\sqrt{9-x^2}}{9x} + C}$$

Example 3:

$$\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{dx}{\sqrt{x^2+1}^{\,3}}$$

$\tan\theta = x$

$dx = \sec^2\theta \, d\theta$

$$\int \frac{\sec^2\theta \, d\theta}{\sqrt{\tan^2\theta + 1}^{\,3}}$$

$$\int \frac{\sec^2\theta}{\sec^3\theta} \, d\theta$$

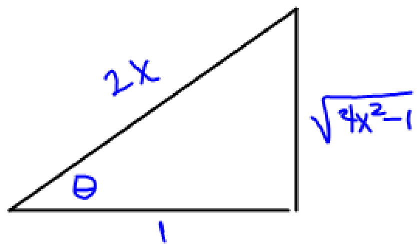
$$\int \cos\theta \, d\theta$$

$\sin\theta + C$

$$\boxed{\frac{x}{\sqrt{x^2+1}} + C}$$

Example 4:

$$\int \frac{dx}{\sqrt{4x^2-1}} = \int \frac{dx}{\sqrt{(2x)^2-1}}$$



$\sec\theta = 2x$

$x = \frac{1}{2}\sec\theta \rightarrow dx = \frac{1}{2}\sec\theta\tan\theta \, d\theta$

$$\frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}} \, d\theta$$

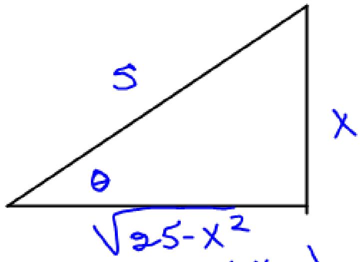
$$\frac{1}{2}\int \sec\theta \, d\theta$$

$\frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$

$$\boxed{\frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right| + C}$$

Example 5:

$$\int \frac{x^2}{\sqrt{25-x^2}}\,dx$$



$\sin\theta = \frac{x}{5} \rightarrow \theta = \arcsin(x/5)$

$x = 5\sin\theta \rightarrow dx = 5\cos\theta\, d\theta$

$$\int \frac{25\sin^2\theta}{\sqrt{25-25\sin^2\theta}} \cdot 5\cos\theta\, d\theta$$

$$25\int \sin^2\theta\, d\theta$$

$$\frac{25}{2}\int 1-\cos(2\theta)\, d\theta$$

$$\frac{25}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

$$\frac{25}{2}\left[\arcsin(x/5) - \sin\theta\cos\theta\right] + C$$

$$\boxed{\frac{25}{2}\left[\arcsin(x/5) - \frac{x\sqrt{25-x^2}}{25}\right] + C}$$

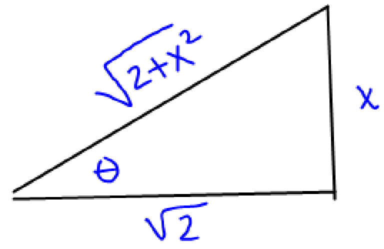
Example 6:

$$\int \frac{1}{4+4x^2+x^4}\,dx = \int \frac{1}{(2+x^2)^2}\,dx$$

$\tan\theta = \frac{x}{\sqrt{2}}$

$x = \sqrt{2}\tan\theta$

$dx = \sqrt{2}\sec^2\theta\, d\theta$



$$\int \frac{1}{(2+2\tan^2\theta)^2} \cdot \sqrt{2}\sec^2\theta\, d\theta$$

$$\frac{\sqrt{2}}{4}\int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$$

$$\frac{\sqrt{2}}{4}\int \cos^2\theta\, d\theta$$

$$\frac{\sqrt{2}}{8}\int 1+\cos(2\theta)\, d\theta$$

$$\frac{\sqrt{2}}{8}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

$$\frac{\sqrt{2}}{8}\left[\arctan(x/\sqrt{2}) + \sin\theta\cos\theta\right] + C$$

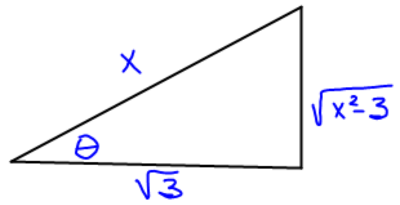
$$\boxed{\frac{\sqrt{2}}{8}\left[\arctan(x/\sqrt{2}) + \frac{x\sqrt{2}}{2+x^2}\right] + C}$$

Example 7:

$$\int x^3\sqrt{x^2-4}\,dx$$

Example 8:

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$$



$$\sec\theta = \frac{x}{\sqrt{3}} \longrightarrow \theta = \operatorname{arcsec}\left(\frac{x}{\sqrt{3}}\right)$$

$$x = \sqrt{3}\sec\theta \longrightarrow dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$$

$$\int_0^{\pi/6} \frac{\sqrt{3\sec^2\theta - 3}}{\cancel{\sqrt{3}\sec\theta}} \cdot \cancel{\sqrt{3}\sec\theta}\tan\theta\,d\theta$$

$$\sqrt{3}\int_0^{\pi/6} \tan^2\theta\,d\theta$$

$$\sqrt{3}\int_0^{\pi/6} \sec^2\theta - 1\,d\theta$$

$$\sqrt{3}\Big[\tan\theta - \theta\Big]_0^{\pi/6}$$

$$\boxed{\sqrt{3}\left[\frac{1}{\sqrt{3}} - \pi/6\right]}$$