Fundamental Theorem of Calculus

If a function f is continuous on the closed interval [a, b] and F is an antiderivative of f on the interval [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This gives us a way to analyze the integrals of functions analytically.

Examples: Calculate the definite integrals given below.

a. $\int_1^6 \underbrace{(x^2 - 2x)}_{f(x)}dx$

$$= \left[\overbrace{\frac{x^3}{3} - x^2 + \underline{C}}^{F(x)}\right]_1^6$$

$$= \underbrace{\left(\frac{6^3}{3} - 6^2 + \cancel{C}\right)}_{F(b)} - \underbrace{\left(\frac{1^3}{3} - 1^2 + \cancel{C}\right)}_{F(a)}$$

$$= \frac{216}{3} - 36 - \frac{1}{3} + 1$$

$$= \frac{215}{3} - 35$$

$$= \frac{110}{3} \approx 36.667$$

b. $\int_1^4 3\sqrt{x}\,dx$

$$3\int_1^4 x^{1/2}\,dx$$

$$\left[3\cdot\frac{x^{3/2}}{3/2}\right]_1^4$$

$$\left[2x^{3/2}\right]_1^4$$

$$2(4)^{3/2}-2(1)^{3/2}$$

$$\boxed{14}$$

c. $\int_0^2 |2x-1|\,dx$

$$|2x-1| = \begin{cases} -(2x-1), & x<1/2 \\ 2x-1, & x>1/2 \end{cases}$$

$$-\int_0^{1/2}(2x-1)\,dx + \int_{1/2}^{2} 2x-1\,dx$$

$$-\left[(x^2-x)\right]_0^{1/2} + \left[(x^2-x)\right]_{1/2}^{2}$$

$$\left[(x-x^2)\right]_0^{1/2} + \left[x^2-x\right]_{1/2}^{2}$$

$$\left[\left(\frac{1}{2}-\left(\frac{1}{2}\right)^2\right)-(0-0)\right]+\left[(2^2-2)-\left(\left(\frac{1}{2}\right)^2-\frac{1}{2}\right)\right]$$

$$\frac{1}{4}+2+\frac{1}{4}$$

$$\boxed{\frac{5}{2}}$$

d. $\int_{-3}^{2} f(x)\,dx,\ f(x) = \begin{cases} -x^2 - 2x - 1, & x \le -1 \\ -x - 1, & x > -1 \end{cases}$

$$\int_{-3}^{-1} -x^2 - 2x - 1\,dx + \int_{-1}^{2} -x - 1\,dx$$

$$\left[-\frac{x^3}{3} - x^2 - x\right]_{-3}^{-1} + \left[-\frac{x^2}{2} - x\right]_{-1}^{2}$$

$$\left[\left(\frac{1}{3} \cancel{-1} \cancel{+1}\right) - \left(\cancel{9} \cancel{-9} + 3\right)\right] + \left[\left(-2-2\right) - \left(-\frac{1}{2} + 1\right)\right]$$

$$-\frac{8}{3} - \frac{9}{2}$$

$$\boxed{\frac{-43}{6}}$$

e. $\int_{0}^{\frac{\pi}{6}} -2\sec x \cdot \tan x\,dx$

$$-2\int_{0}^{\pi/6} \sec x \tan x\,dx$$

$$\left[-2\sec x\right]_{0}^{\pi/6}$$

$$\left(-2 \cdot \frac{2}{\sqrt{3}}\right) - \left(-2\right)$$

$$\boxed{\frac{-4}{\sqrt{3}} + 2}$$

f. $\displaystyle\int_{-6}^{1} \left| x^2 + 5x \right| \, dx$