Definite Integrals

A ***definite integral*** is the area under a specified region of a curve.

$\lim_{\Delta x \to 0}$

SECANT: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$ $\frac{dy}{dx}$

Definition of definite Integral

If f is defined and continuous on the closed interval [a, b] and

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$$

exits, then f is ***integrable*** on [a , b]. The ***definite integral*** is represented by:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$$

Where the number ***a*** is the ***lower limit of integration*** and the number ***b*** is the ***upper limit of integration***.

Definite integrals have the same properties as indefinite integrals. There are a few others that we need to be aware of:

1. $\int_a^a f(x)dx = 0$

2. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

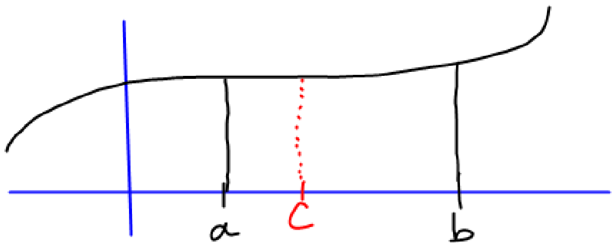
3. If a < c < b, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

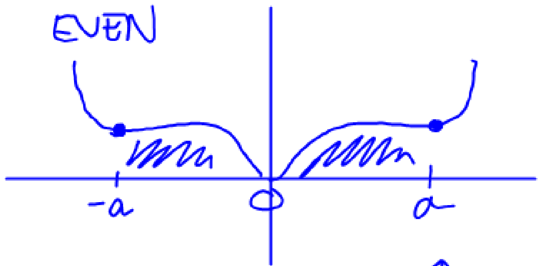
4. Given that $f(x)$ is an even function, $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx = 2\int_{-a}^0 f(x)dx$

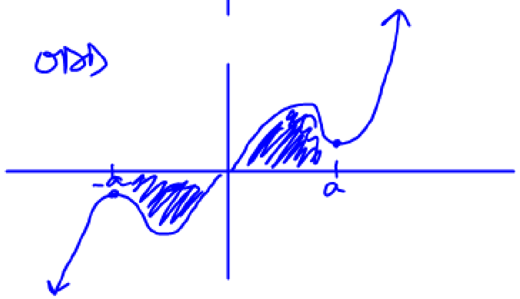
Symmetric about y-Axis

5. Given that $f(x)$ is an odd function, $\int_{-a}^a f(x)dx = 0$

180° Rotational Symmetry about origin







Evaluating Definite Integrals

The Riemann sums that we studied can be used to approximate the definite integrals.

1. Approximate $\int_0^{\pi}(2x\sin x)dx$ using four subintervals of equal length and a Right Hand Riemann sum.

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$\int_0^{\pi}(2x\sin x)dx \approx \frac{\pi}{4}\left[f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi) \right]$$

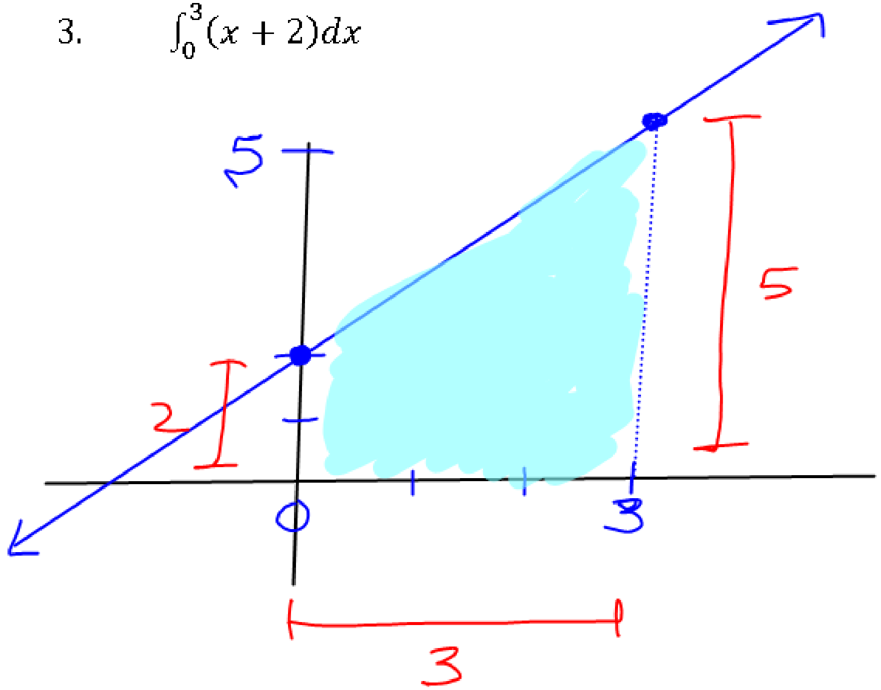
2. Given the table to the right, approximate $\int_{-2}^{9} P(x)dx$ using six subintervals and a Trapezoidal sum.

x	-2	0	1	3	5	8	9
$P(x)$	5	8	2	-4	-1	2	5

$$\int_{-2}^{9} P(x)\,dx \approx \frac{1}{2}\left[(5+8)\cdot 2 + (8+2)\cdot 1 + (2+-4)2 + (-4+-1)2 + (-1+2)3 + (2+5)1\right]$$

- Some definite integrals can be evaluated using prior geometric knowledge.

3. $\int_0^3 (x+2)\,dx$



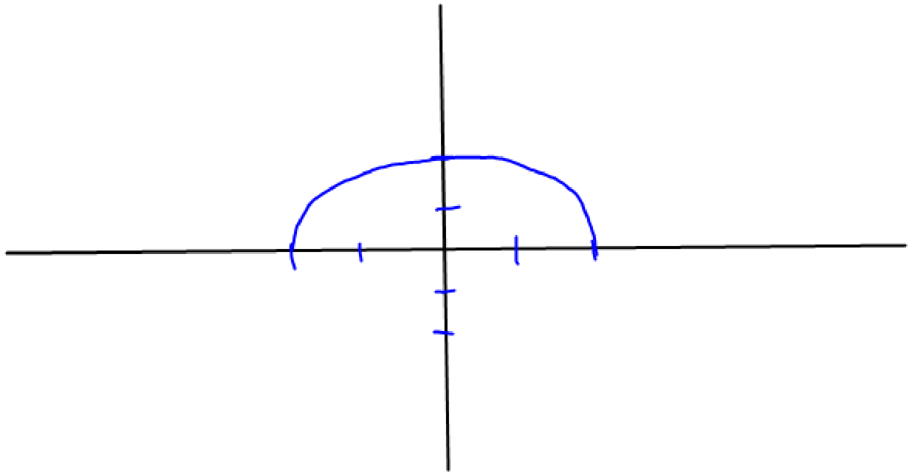
$$\int_0^3 (x+2)\,dx = \frac{1}{2}(2+5)\cdot 3$$
$$= \frac{21}{2}$$

4. $\int_{-2}^{2} \sqrt{4 - x^2}\,dx = \frac{1}{2}\pi(2)^2 = 2\pi$

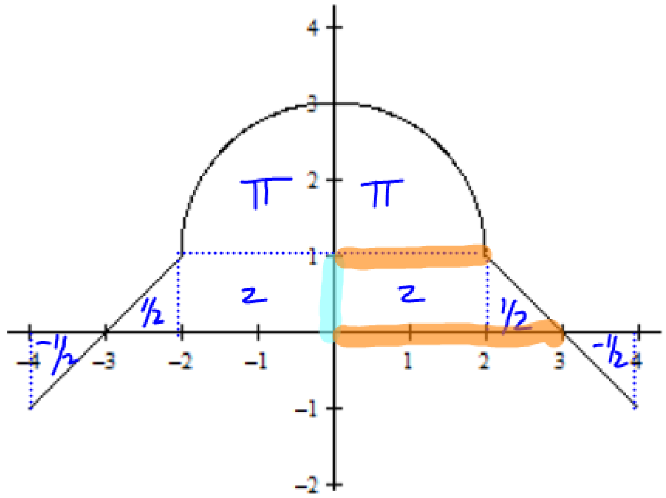
$y^2 = \sqrt{4-x^2}^{\,2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$



5. Pictured to the right is the graph of the function f(x). Identify the following.



a. $\int_0^3 f(x)dx = \pi + \frac{5}{2}$

b. $\int_0^4 f(x)dx = \pi + 2$

c. $\int_{-3}^3 f(x)dx = 2\int_0^3 f(x)dx$

$= 2\left[\pi + \frac{5}{2}\right] = 2\pi + 5$

-We can apply the limit process to evaluate definite integrals as well.

6. Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$

-What about functions that don't create a geometric shape on the graph?

Antidifferentiation and Integrals

Evaluating a definite integral requires the use of the **antiderivative.** An antiderivative reverses the differentiation process.

Example 1: Find a function F whose derivative is $f(x) = 3x^2$.

$F(x) = x^3 \rightarrow$ Antiderivative of $3x^2$

Consider the functions: $F(x) = x^3$ $G(x) = x^3 - 5$ $H(x) = x^3 + 10$

$F'(x) = 3x^2$ $G'(x) = 3x^2$ $H'(x) = 3x^2$

Therefore when finding general antiderivatives, you must include the ***constant of integration.*** By including this constant, you are identifying the family of functions that represent all antiderivative of the given function. Finding a general antiderivative is finding a solution to a ***differential equation.***

$+ C$

Notation for Antiderivatives

Given the differential equation of the form $\boxed{\frac{dy}{dx} = f(x)}$ we can rewrite this is an equivalent form:

$$\underline{\int \boldsymbol{dy} = \int \boldsymbol{f(x)dx}} \quad \rightarrow \quad \underline{\boldsymbol{y} = \int \boldsymbol{f(x)dx} = \boldsymbol{F(x) + c}}$$

This notation is called the ***indefinite integral.*** The expression $\int f(x)dx$ is read as *the antiderivative of f with respect to x.* The terms indefinite integral and antiderivative are synonymous.

Integration is the "inverse" of differentiation. Therefore:

$$\int F'(x)dx = F(x) + C \qquad \text{And} \qquad \frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

*For a full list of the properties of integrals, see page 250.

The biggest rule when evaluating integrals is the power rule for integration. $= \int x^{-1}\,dx$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1 \rightarrow = \int \frac{1}{x}\,dx$$

* Involves Natural Log

Examples: Evaluate each indefinite integral.

a. $\int 3x\,dx$

$3 \cdot \int x^1\,dx$

$3 \cdot \frac{x^{1+1}}{1+1} + C$

$\boxed{\frac{3}{2}x^2 + C}$

b. $\int \frac{1}{x^3}\,dx$

$\int x^{-3}\,dx$

$\frac{x^{-3+1}}{-3+1} + C$

$\boxed{-\frac{1}{2x^2} + C}$

c. $\int \sqrt{x}\, dx$

$$\int x^{1/2}\, dx$$

$$\frac{x^{1/2+1}}{\frac{1}{2}+1} + C$$

$$\frac{x^{3/2}}{\frac{3}{2}} + C$$

$$\boxed{\frac{2}{3}x^{3/2} + C}$$

d. $\int (3x^4 - 5x^2 + x)\, dx$

$$3\int x^4 dx - 5\int x^2 dx + \int x\, dx$$

+C +C +C

$$3 \cdot \frac{x^5}{5} - 5 \cdot \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\boxed{\frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C}$$

e. $\int \frac{x+1}{\sqrt{x}}\,dx$

$$\int \frac{x^1}{x^{1/2}} + \frac{1}{x^{1/2}}\,dx$$

$$\int x^{1/2} + x^{-1/2}\,dx$$

$$\frac{x^{1/2+1}}{1/2+1} + \frac{x^{-1/2+1}}{-1/2+1} + C$$

$$\boxed{\frac{2}{3}x^{3/2} + 2x^{1/2} + C}$$

f. $\int 2\sin x\,dx$

$$\boxed{-2\cos x + C}$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

g. $\int dx$ $\qquad \int 2\,dx$ $\qquad \int -3\,dt$

$x + C$ $\qquad 2x + C$ $\qquad -3t + C$

$$\int x^0\,dx$$

If an ***initial condition*** is provided, a ***particular solution*** can be determined for the integral of a function.

Examples:

a. Find the particular solution of the differential equation $F'(x) = \frac{1}{x^2}$, $x > 0$ that satisfies the initial condition $F(1) = 0$.

$$\int F'(x)\,dx = F(x) + C$$

$$\int x^{-2}\,dx = -\frac{1}{x} + C$$

$$\frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1}$$

$$F(x) = -\frac{1}{x} + C$$

$$0 = -\frac{1}{1} + C$$

$$1 = C$$

$$\boxed{F(x) = -\frac{1}{x} + 1}$$

$y(x)$

b. If $\frac{dy}{dx} = \cos x$ and $\boxed{y(0) = -1}$, what would be a particular solution?

$\cancel{dx} \cdot \frac{dy}{\cancel{dx}} = \cos x \cdot dx$

$\int dy = \int \cos x \, dx$

$y = \sin x + C \longrightarrow \boxed{y = \sin x - 1}$

$-1 = \sin(0) + C$

$-1 = C$

c. Evaluate $\int (x+1)(3x-2)dx$ if you know that the point $(2,10)$ exists on the antiderivative.

$$\int (3x^2+x-2)dx = x^3+\tfrac{1}{2}x^2-2x + C$$

$$10 = (2)^3+\tfrac{1}{2}(2)^2-2(2)+C$$

$$10 = 6+C$$

$$4 = C$$

$$\boxed{x^3+\tfrac{1}{2}x^2-2x+4}$$