Riemann Sums Continued

Formal Definitions for Left and Right Endpoint Riemann Sums

Let f be continuous and nonnegative on the interval [a,b].

Left Endpoint:

$$\sum_{i=1}^{n} f(m_i) \cdot \Delta x$$

Right Endpoint

$$\sum_{i=1}^{n} f(M_i) \cdot \Delta x$$

– Δx: The change in the x value. Represents the width of each rectangle. $\Delta x = \frac{b-a}{n}$

– m_i: Left (lower) endpoint. $m_i = a + (i-1)\Delta x$

– M_i: Right (upper) endpoint. $M_i = a + i\Delta x$

Common Summation Formulas

1. $$\sum_{i=1}^{n} c = cn$$

2. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

3. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

4. $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 1: Find the left endpoint Riemann Sum using n subintervals of equal length for the function $y = x^2$ on the interval [0, 4]. Find the approximate area using 6 subintervals.

$\Delta x = \frac{2}{3}$ $\quad m_i = 0 + (i-1)\cdot\frac{2}{3} = (i-1)\cdot\frac{2}{3}$

$$\sum_{i=1}^{n} f(m_i)\cdot\Delta x$$

$$\sum_{i=1}^{6} f\left((i-1)\cdot\tfrac{2}{3}\right)\cdot\frac{2}{3}$$

$$\sum_{i=1}^{6}\left[(i-1)\cdot\tfrac{2}{3}\right]^2\cdot\frac{2}{3}$$

$$\frac{8}{27}\sum_{i=1}^{6}(i-1)^2 = \frac{8}{27}\sum_{i=1}^{6} i^2 - 2i + 1$$

$\frac{n(n+1)(2n+1)}{6}$ $\quad \frac{n(n+1)}{2}$ $\quad n$

$$= \frac{8}{27}\left[\frac{\cancel{6}(7)(13)}{\cancel{6}} - \cancel{2}\cdot\frac{6(7)}{\cancel{2}} + 6\right]$$

$$= \frac{8}{27}\left[91 - 42 + 6\right] = \frac{8}{27}\left[55\right] = \frac{440}{27}$$

$$\text{AREA} \approx \frac{2}{3}\left[f(0)+f\left(\tfrac{2}{3}\right)+f\left(\tfrac{4}{3}\right)+f(2)+f\left(\tfrac{8}{3}\right)+f\left(\tfrac{10}{3}\right)\right]$$

$$\approx \frac{2}{3}\left[0 + \frac{4}{9} + \frac{16}{9} + 4 + \frac{64}{9} + \frac{100}{9}\right]$$

$$\approx \frac{2}{3}\left[\frac{220}{9}\right] = \frac{440}{27} = 16.296$$

$f(x) = x^2$

$f(☺) = (☺)^2$

Example 2: Find the right endpoint Riemann Sum using n subintervals of equal length for the function $y = x^3$ on the interval [-1, 5]. Find the approximate area using 4 subintervals.

$$\Delta x = \frac{5 - -1}{4} = \frac{6}{4} = \frac{3}{2} \qquad M_i = -1 + \frac{3}{2}i$$

$$\sum_{i=1}^{4} f\left(-1 + \frac{3}{2}i\right) \cdot \frac{3}{2}$$

$$\sum_{i=1}^{4} \left(\frac{3}{2}i - 1\right)^3 \cdot \frac{3}{2}$$

$$\frac{3}{2} \sum_{i=1}^{4} \left(\frac{3}{2}i\right)^3 - 3\left(\frac{3}{2}i\right)^2(1)^1 + 3\left(\frac{3}{2}i\right)(1)^2 - 1$$

$$\frac{3}{2} \sum_{i=1}^{4} \frac{27}{8}i^3 - \frac{27}{4}i^2 + \frac{9}{2}i - 1$$

$$\frac{3}{2}\left[\frac{27}{\cancel{8}_{2}}\left(\frac{\cancel{4^2}^{2} \cdot 5^2}{\cancel{4}}\right) - \frac{27}{\cancel{4}}\left(\frac{\cancel{4}(5)(\cancel{9}^{3})}{\cancel{6}\ 2}\right) + \frac{9}{\cancel{2}}\left(\frac{\cancel{4}(5)}{\cancel{2}}\right) - 1(4)\right]$$

$$\frac{3}{2}\left[\frac{675}{2} - \frac{405}{2} + 45 - 4\right] = \boxed{264}$$

Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x – axis, and the vertical lines $x = a$ and $x = b$ is

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

Where $\boxed{c_i}$ is either the left or right endpoint and $\Delta x = (b - a)/n$.

Simpler to use right endpoints

Example 3: Find the area of the region bounded by the graph of $y = x^2$ on the interval [0, 2].

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$c_i = 0 + i \cdot \frac{2}{n} = \frac{2}{n} i$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2}{n} i\right) \cdot \frac{2}{n}$$

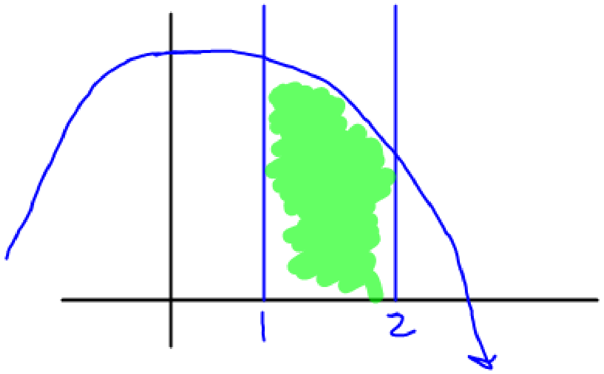
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n} i\right)^2 \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \boxed{\sum_{i=1}^{n} i^2}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{(2n^3 + 3n^2 + n)}{6} = \frac{16}{6} = \boxed{\frac{8}{3}}$$

Example 4: Find the area of the region bounded by the graph of $f(x) = 4 - x^2$, the x – axis, and the vertical lines $x = 1$ and $x = 2$.



$\Delta x = \frac{2-1}{n} = \frac{1}{n}$ $\quad c_i = 1 + \frac{1}{n} i$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{1}{n}i\right)\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \left(1+\frac{1}{n}i\right)^2\right]\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \left(\frac{1}{n^2}i^2 + \frac{2}{n}i + 1\right)\right]\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[3 - \frac{1}{n^2}i^2 - \frac{2}{n}i\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \left[3n - \frac{1}{n^2}\left(\frac{2n^3+3n^2+n}{6}\right) - \frac{2}{n}\left(\frac{n^2+n}{2}\right)\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \left[3 - \frac{2n^3+3n^2+n}{6n^3} - \frac{2n^2+2n}{2n^2}\right]$$

$$= 3 - \frac{1}{3} - 1$$

$$= \frac{5}{3}$$

Example 5: Find the area of the region bounded by the graph of $f(x) = 2x - x^3$, the x – axis, and the vertical lines $x = 0$ and $x = 1$.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n} \qquad c_i = 0 + i \cdot \frac{1}{n} = i \cdot \frac{1}{n}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(i \cdot \frac{1}{n}\right) - \left(i \cdot \frac{1}{n}\right)^3 \right] \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2}{n} i - \frac{1}{n^3} i^3 \right] \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \left[\frac{2}{n}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^3} \cdot \frac{n^2(n+1)^2}{4} \right] \frac{1}{n}$$

$$= \lim_{n\to\infty} \left[\frac{2n^2+2n}{2n^2} - \frac{n^4+2n^3+n^2}{4n^4} \right]$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

~~**Example 6:** Find the area of the region bounded by the graph of $f(x) = 2x^3 - x + 4$ and the x-axis on the interval [1, 3].~~