At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$

$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$

$3(G - 27)^{1/3} = -t + C$

$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$

$3(G - 27)^{1/3} = 12 - t$

$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

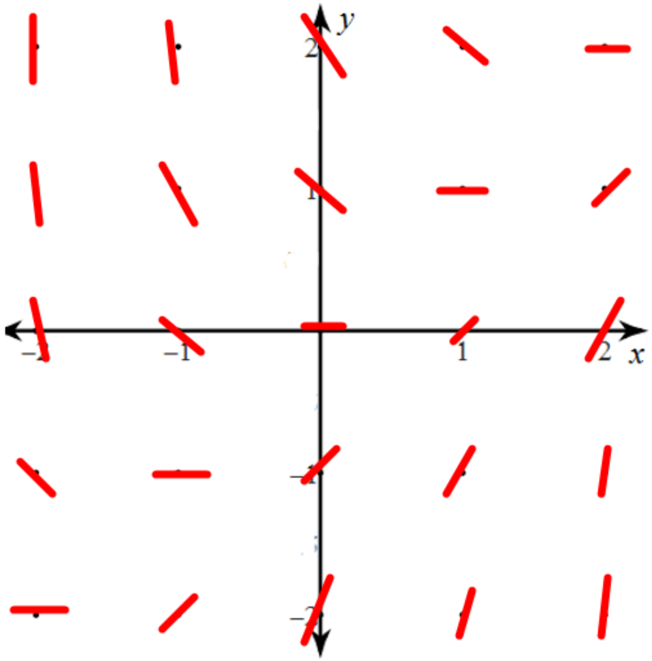
Slope Fields

Sometimes, solving a differential equation analytically can be impossible. There is a graphical approach that can be used to learn a lot about the solution of a differential equation.

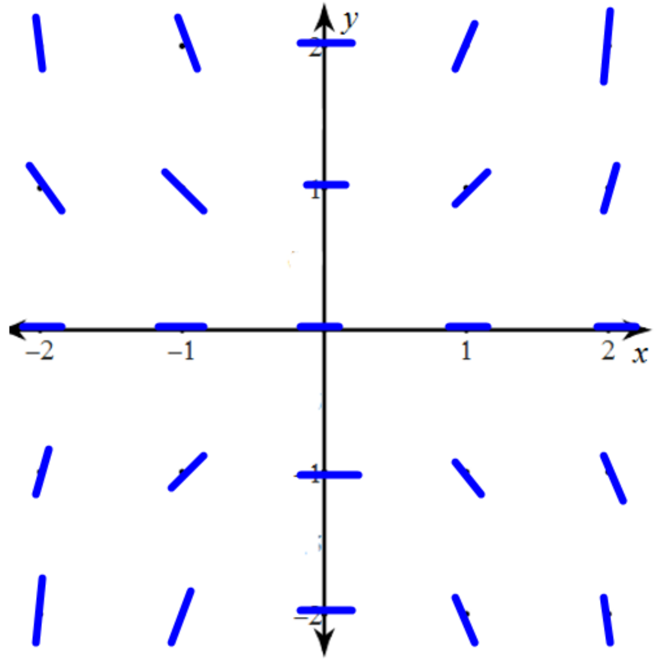
- Consider a differential equation of the form: $y' = F(x, y)$
    - At every point (x, y) in the plane, the differential equation determines the **slope** of the **solution** at that point.
    - At each point, a short **line segment** can be drawn that corresponds to the **slope** of the solution curve through that point.
    - A slope field displays the general shape of **all possible solutions.**

Example 1: Sketch a slope field for the differential equation on the given graph.

a. $y' = x - y$



b. $y' = xy$



$\frac{dy}{dx} = xy$

$\frac{1}{y}\,dy = x\,dx$

$\ln|y| = \frac{1}{2}x^2 + C$

$e \qquad e$

$|y| = Ce^{\frac{1}{2}x^2}$

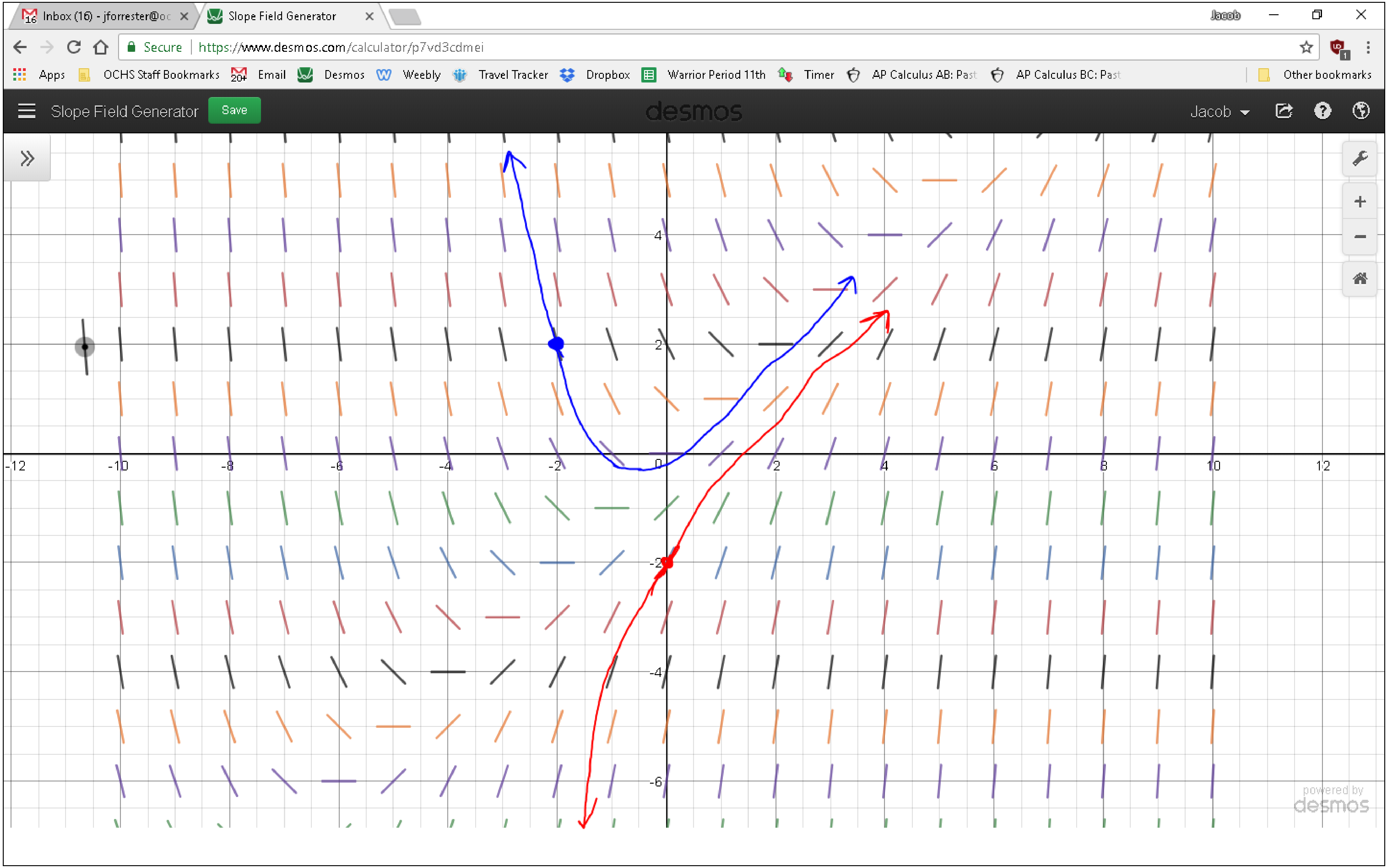
$y = e^{\frac{1}{2}x^2}$

$x^2 \cdot x^3$

x^{2+3}

$e^{\frac{1}{2}x^2 + C}$

$e^{\frac{1}{2}x^2} \cdot e^C$



Inbox (16) - jforrester@oc
Slope Field Generator
Jacob
Secure
https://www.desmos.com/calculator/p7vd3cdmei
Apps
OCHS Staff Bookmarks
Email
Desmos
Weebly
Travel Tracker
Dropbox
Warrior Period 11th
Timer
AP Calculus AB: Past
AP Calculus BC: Past
Other bookmarks
Slope Field Generator
Save
desmos
Jacob
-12
-10
-8
-6
-4
-2
0
2
4
6
8
10
12
4
2
-2
-4
-6
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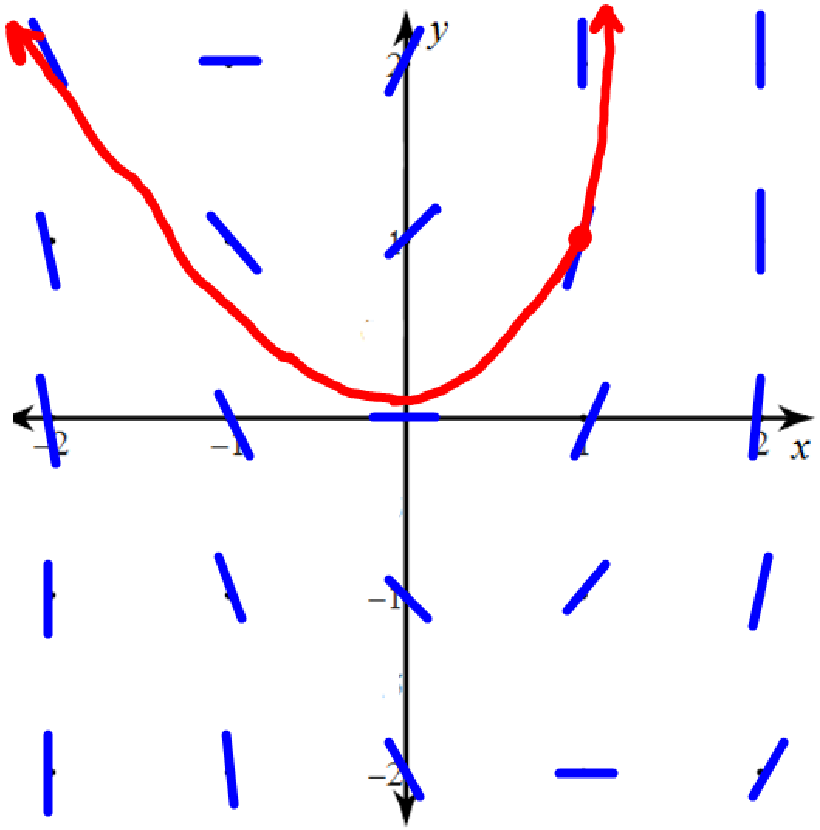
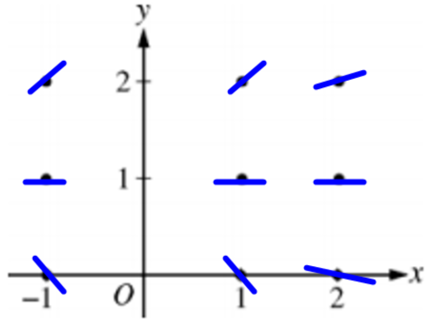
Example 2: Sketch a slope field for the differential equation $y' = 2x + y$. Use the slope field to sketch the solution that passes through the point $(1, 1)$.



Example 4: **(2008 AB Exam Question 5)** Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

a. On the axes provided, sketch a slope field for the given differential equation the nine points indicated.



$\frac{dy}{dx} = \frac{y-1}{x^2}$

b. Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

c. For the particular solution $y = f(x)$ described in part b, find $\lim_{x\to\infty} f(x)$.

(b) $\int \frac{1}{y-1}\,dy = \int \frac{1}{x^2}\,dx$

$u = y-1$, $du = dy$

$= \int x^{-2}\,dx$

$\int \frac{1}{u}\,du = \frac{x^{-1}}{-1} + C$

$\ln|u|$

$\ln|y-1| = -\frac{1}{x} + C$

Use $(2,0)$

$\ln|-1| = \frac{-1}{2} + c$

$0 = -\frac{1}{2} + c$

$\frac{1}{2} = c$

$e^{\ln|y-1|} = e^{-\frac{1}{x}+\frac{1}{2}}$

$|y-1| = e^{-\frac{1}{x}+\frac{1}{2}}$

$y-1 = \pm e^{-\frac{1}{x}+\frac{1}{2}}$

$y = \pm e^{-\frac{1}{x}+\frac{1}{2}} + 1$

Since $f(2) = 0$, solution is

$y = -e^{-1/x+1/2} + 1$

(c) $\lim_{x\to\infty} -e^{-1/x+1/2} + 1$

Since $\lim_{x\to\infty} -\frac{1}{x} = 0$

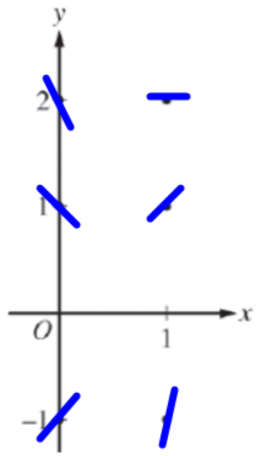
$\Downarrow$

$\lim_{x\to\infty} -e^{-1/x+1/2} + 1 = -e^{1/2} + 1$

$= 1 - \sqrt{e}$

Example 5: **(2015 AB Exam Question 4)** Consider the differential equation $\frac{dy}{dx} = 2x - y$.

a. On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



$\frac{dy}{dx} = 2x - y$

b. Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

c. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

d. Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$

$= 2 - (2x - y)$

$= 2 - 2x + y$

In Quad II, $x < 0$ & $y > 0$,

therefore $\frac{d^2y}{dx^2} = 2 - 2x + y > 0$.

So all solution curves in Quad II are concave up.

(c) $\left.\frac{dy}{dx}\right|_{x=2} = 2(2) - 3 = 1$

Since $\left.\frac{dy}{dx}\right|_{x=2} \neq 0$,

f has neither a relative minimum or maximum at $x = 2$.

(c) If $y = mx + b$ is a solution, then $\frac{dy}{dx}$ = constant $= m$.

Therefore, $\frac{d^2y}{dx^2} = 0$ for all (x, y).

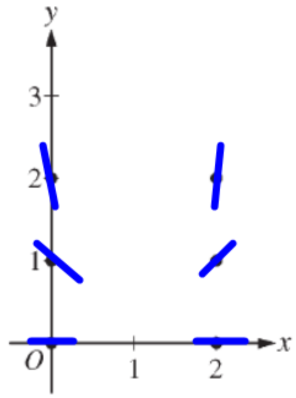
$2 - 2x + y = 0 \Rightarrow y = 2x - 2$

So, $m = 2$ & $b = -2$.

2016 Question 4

Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



$\frac{dy}{dx} = \frac{y^2}{x-1}$

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

(b) $\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{9}{2-1} = 9$

$y - 3 = 9(x-2)$

$y = 9(x-2) + 3$

$f(2.1) \approx 9(2.1-2) + 3 = 3.9$

(c) $\int \frac{1}{y^2}\, dy = \int \frac{1}{x-1}\, dx$ $u = x-1$, $du = dx$

$\int y^{-2}\, dy = \int \frac{1}{u}\, du$

$\frac{y^{-1}}{-1} = \ln|u| + c$

$-\frac{1}{y} = \ln|x-1| + c$

Using $(2,3)$

$-\frac{1}{3} = \ln|1| + c$

$-\frac{1}{3} = c$

$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$

$\frac{1}{y} = -\ln|x-1| + \frac{1}{3}$

$y = \frac{1}{-\ln|x-1| + \frac{1}{3}}$

or

$y = \frac{-1}{\ln|x-1| - \frac{1}{3}}$