Separable Differential Equations

A separable differential equation is one in which all *x* terms can be collect on one side with *dx* and all *y* terms can be collected on the other side with *dy*. When this is the case, we can use the method of **separation of variables** to solve a differential equation.

Separable Differential Equations:

$x^2 + 3y\frac{dy}{dx} = 0$

$(\sin x)y' = \cos x$

$\frac{xy'}{e^{y+1}} = 2$

Non-Separable Differential Equations:

$\frac{dy}{dx} = x + y$

$2yy' + 2y = \frac{x^2}{2} + x$

-When solving, **group everything with y** on one side and **everything with x** on the other and then **integrate both sides**.

Example 1: Find the general solution of $(x^2+4)\frac{dy}{dx} = xy$.

$(x^2+4)\,dy = xy \cdot dx$

$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+4}\,dx$

$u = x^2+4$

$du = 2x\,dx$

$\ln|y| + C = \frac{1}{2}\int \frac{1}{u}\,du$

$\ln|y| + C_1 = \frac{1}{2}\ln|u| + C_2$

$e^{\ln|y|} = e^{\frac{1}{2}\ln|x^2+4| + C} \rightarrow$ A solution

$|y| = e^{\frac{1}{2}\ln|x^2+4| + C}$

$|x| = 2$

$\Downarrow$

$x = \pm 2$

$y = \pm e^{\frac{1}{2}\ln|x^2+4| + C} \rightarrow$ A solution $y = f(x)$

$y = \pm e^{\ln\sqrt{x^2+4}} \cdot e^{C}$

$y = \pm C\sqrt{x^2+4}$

Example 2: Given the initial condition $\boxed{y(0)=1}$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}(y^2-1)dy = 0$$

$$\frac{e^{-x^2}(y^2-1)dy}{y\cdot e^{-x^2}} = \frac{-xy\,dx}{y\cdot e^{-x^2}}$$

$$\int \frac{y^2-1}{y}dy = \int -xe^{x^2}dx$$

$u = x^2$
$du = 2x\,dx$

$$\int y - \frac{1}{y}\,dy = -\frac{1}{2}\int e^u\,du$$

$$\frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^u + C$$

$$\frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + C \longrightarrow \boxed{\frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + 1}$$

$$\frac{1}{2}(1)^2 - \ln(1) = -\frac{1}{2}e^{0^2} + C$$

$$\frac{1}{2} = -\frac{1}{2} + C$$

$$1 = C$$

Example 3: Find the equation of the curve that passes through the point (1, 3) and $\boxed{\text{has a slope of } y/x^2}$ at any point (x, y).

$$\frac{dy}{dx} = \frac{y}{x^2}$$

$$x^2\,dy = y\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx \longrightarrow \int x^{-2}\,dx$$

$$\frac{x^{-2+1}}{-2+1}$$

$$\ln(3) = -1 + C \longleftarrow \ln|y| = \frac{-1}{x} + C$$

$$\ln(3) + 1 = C$$

$$e^{\ln|y|} = e^{-\frac{1}{x} + \ln(3) + 1}$$

$$y = \pm e^{-1/x + \ln(3) + 1}$$

Since (1, 3) is the initial condition, the solution is

$$y = e^{-1/x + \ln(3) + 1}$$

$$y = 3e^{-1/x + 1}$$

Example 4: **(2011 AB Exam Question 5)** At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that $\boxed{W}$ will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measure in tons, and t is measured in years from the start of 2010.

(x, y)
(t, w)

a. Use the line tangent to the graph of W at $\boxed{t=0}$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time t = ¼)

b. Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part(a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time t = ¼.

c. Find the particular solution W = W(t) to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with the initial condition W(0) = 1400.

(a) $(0, 1400)$

$y - y_1 = \frac{dy}{dx}(x - x_1)$

$\left.\frac{dw}{dt}\right|_{t=0} = \frac{1}{25}(1400 - 300) = 44$

$w - 1400 = 44(t - 0)$

$w = 44t + 1400$

$w(1/4) = 44(1/4) + 1400 = 1411$ TONS

(b) $\frac{d^2w}{dt^2} = \frac{d}{dt}\left[\frac{1}{25}(w - 300)\right] = \frac{1}{25} \cdot \frac{dw}{dt} = \frac{1}{625}(w - 300)$

Since w is increasing, $w > 1400$. Therefore $\frac{d^2w}{dt^2} > 0$.

Since $\frac{d^2w}{dt^2} > 0$, the estimate in a. is an underestimate.

(c) $\frac{dw}{dt} = \frac{1}{25}(w - 300)$

$\int \frac{1}{w - 300}\, dw = \int \frac{1}{25}\, dt$

$u = w - 300$
$du = dw$

$\int \frac{1}{u}\, du = \frac{1}{25}t + C$

$\ln|u| = \frac{1}{25}t + C$

$\ln|w - 300| = \frac{1}{25}t + C$

$\ln(1100) = C$

$e^{\ln|w - 300|} = e^{\frac{1}{25}t + \ln(1100)}$

$w - 300 = e^{\frac{1}{25}t + \ln(1100)}$

$\boxed{w(t) = 1100e^{\frac{1}{25}t} + 300}$

(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Example 5: **(2013 AB Exam Question 6)**

Consider the differential equation $\frac{dy}{dx} = e^y\left(3x^2 - 6x\right)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

(a) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

Example 6: **(2012 AB Exam Question 5)**

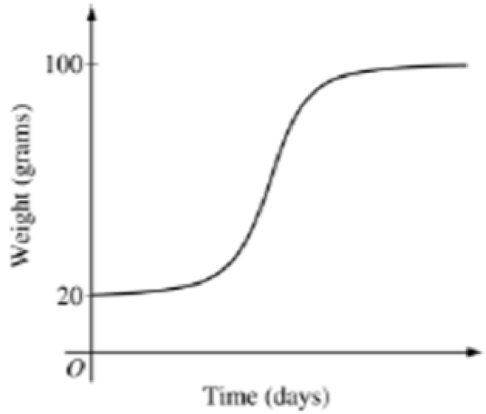
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.