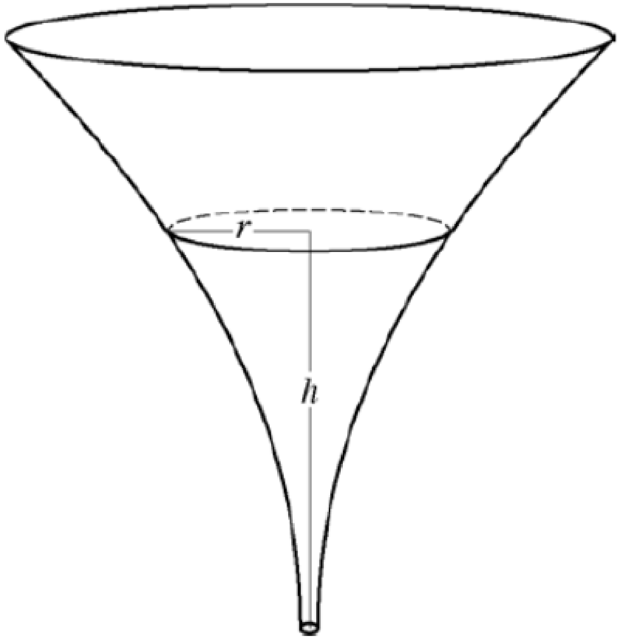
2016 Question 5 Non-Calculator



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

(a) Average radius $= \frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right) dh = \frac{1}{200}\left[3h + \frac{h^3}{3}\right]_0^{10}$

$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - 0\right) = \frac{109}{60}$ in

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi\int_0^{10} \left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2 dh = \frac{\pi}{400}\int_0^{10}\left(9 + 6h^2 + h^4\right) dh$

$= \frac{\pi}{400}\left[9h + 2h^3 + \frac{h^5}{5}\right]_0^{10}$

$= \frac{\pi}{400}\left(\left(90 + 2000 + \frac{100000}{5}\right) - 0\right) = \frac{2209\pi}{40}$ in^3

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$

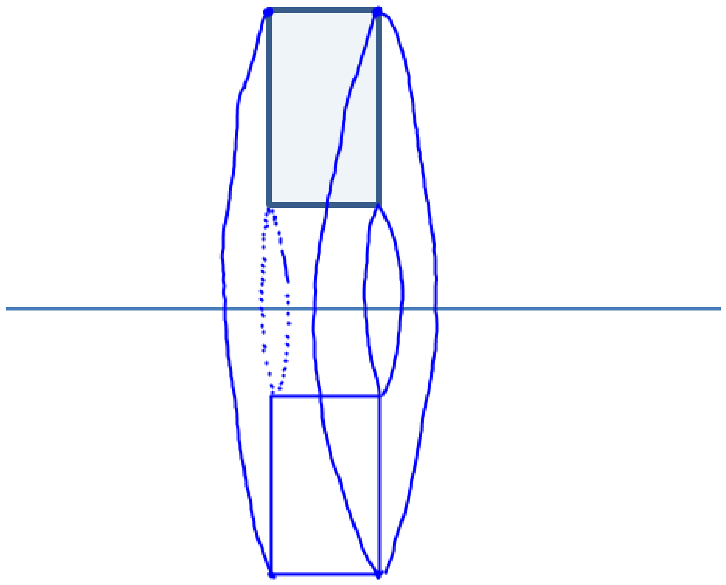
$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{1}{5}\cdot\frac{10}{3} = -\frac{2}{3}$ in/sec

$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

Volume of Revolution: Washer Method

When forming a solid of revolution using the disk method, the area of the region must be adjacent to the axis of revolution. If there is any space between the axis and the region, we need to modify our approach. When there is space, we use the **washer method.**



$$V = \pi(R)^2 \cdot h - \pi(r)^2 \cdot h$$

$$= \pi\left[R^2 - r^2\right]h$$

Consider revolving a region around the x-axis. The larger (or upper) function will determine the **outer radius** while the smaller function (or lower) will determine the **inner radius.**

The volume can be calculated by:

Horizontal Axis of Revolution:

$$\pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx$$

Where f(x) represents the **top function** and g(x) represent the **bottom function.**

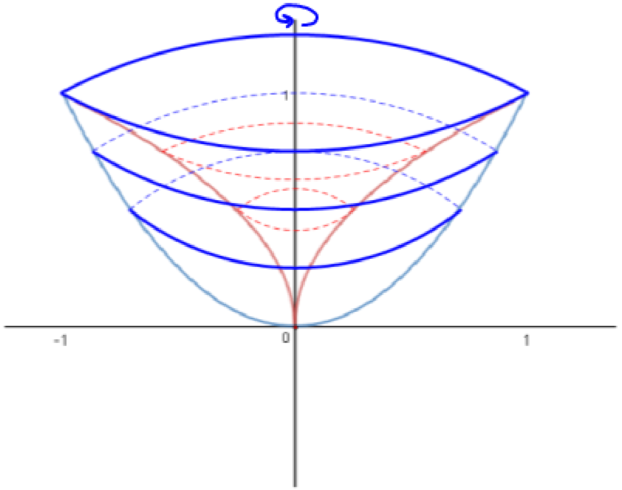
Vertical Axis of Revolution:

$$\pi \int_a^b ([f(y)]^2 - [g(y)]^2)dy$$

Where f(x) represents the **right function** and g(x) represent the **left function.**

<u>Example:</u> Find the volume of the solid formed by revolving the region bound by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the y-axis. Find the volume of the solid by revolving the given region about the line x = 1.

y- axis



$\pi \int_0^1 [\sqrt{y}]^2 - [y^2]^2 \, dy$

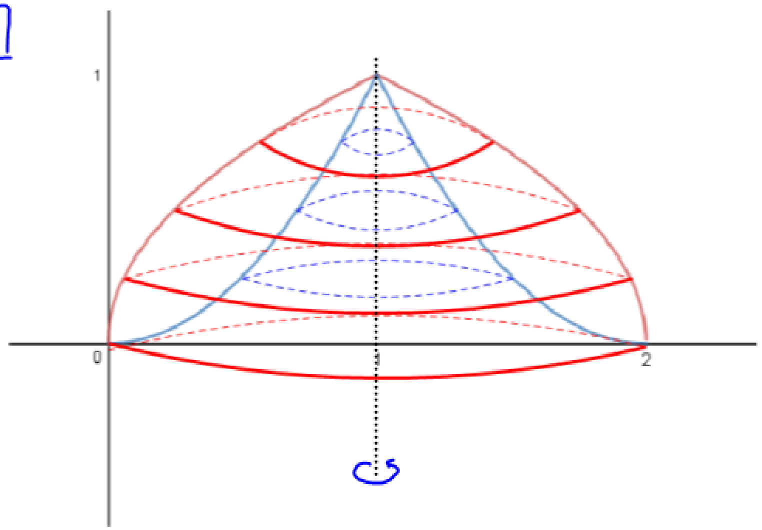
$\pi \int_0^1 y - y^4 \, dy$

$\pi \left[\frac{1}{2}y^2 - \frac{1}{5}y^5\right]_0^1$

$\pi \left[\left(\frac{1}{2} - \frac{1}{5}\right) - (0)\right]$

$\boxed{\frac{3\pi}{10}}$

x = 1



$\pi \int_0^1 [1 - y^2]^2 - [1 - \sqrt{y}]^2 \, dy$

$\pi \int_0^1 1 - 2y^2 + y^4 - [1 - 2\sqrt{y} + y] \, dy$

$\pi \int_0^1 y^4 - 2y^2 - y + 2\sqrt{y} \, dy$

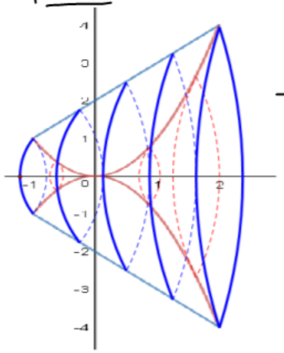
$\pi \left[\frac{1}{5}y^5 - \frac{2}{3}y^3 - \frac{1}{2}y^2 + \frac{4}{3}y^{3/2}\right]_0^1$

$\pi \left[\left(\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3}\right) - (0)\right]$

$\boxed{\frac{11\pi}{30}}$

Example: Find the volume of the solid formed by revolving the region bound by the graphs of $y = x^2$ and $y = x + 2$ about the x-axis. Find the volume of the solid by revolving the given region about the line $y = -1$.

$\boxed{\text{X-AXIS}}$



$$\pi \int_{-1}^{2} [x+2]^2 - [x^2]^2 \, dx$$

$$\pi \int_{-1}^{2} x^2+4x+4 - x^4 \, dx$$

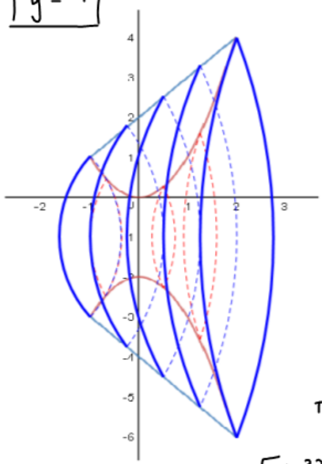
$$\pi \left[\tfrac{1}{3}x^3 + 2x^2 + 4x - \tfrac{1}{5}x^5 \right]_{-1}^{2}$$

$$\pi \left[\left(\tfrac{8}{3} + 8 + 8 - \tfrac{1}{5}\right) - \left(-\tfrac{1}{3} + 2 - 4 + \tfrac{1}{5}\right) \right]$$

$$\pi \left[\left(\tfrac{277}{15}\right) - \left(-\tfrac{32}{15}\right) \right]$$

$$\boxed{\frac{103\pi}{5}}$$

$\boxed{y = -1}$



$$\pi \int_{-1}^{2} [x+2-(-1)]^2 - [x^2-(-1)]^2 \, dx$$

$$\pi \int_{-1}^{2} [x+3]^2 - [x^2+1]^2 \, dx$$

$$\pi \int_{-1}^{2} x^2+6x+9 - (x^4+2x^2+1) \, dx$$

$$\pi \int_{-1}^{2} -x^4 - x^2 + 6x + 8 \, dx$$

$$\pi \left[-\tfrac{1}{5}x^5 - \tfrac{1}{3}x^3 + 3x^2 + 8x \right]_{-1}^{2}$$

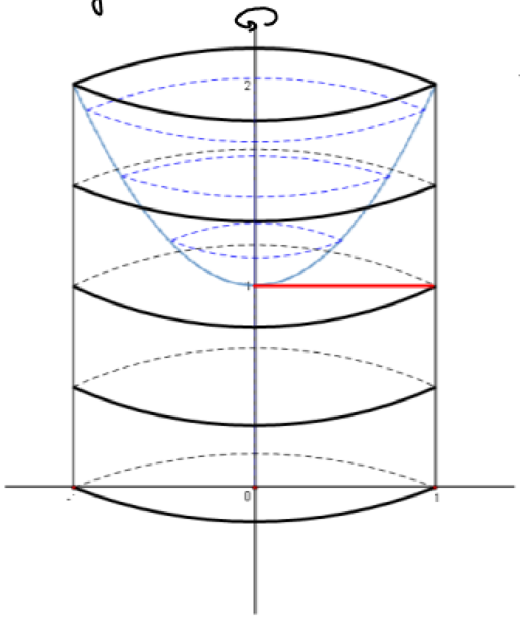
$$\pi \left[\left(-\tfrac{32}{5} - \tfrac{8}{3} + 12 + 16\right) - \left(\tfrac{1}{5} + \tfrac{1}{3} + 3 - 8\right) \right]$$

$$\pi \left[\left(\tfrac{284}{15}\right) - \left(-\tfrac{67}{15}\right) \right]$$

$$\boxed{\frac{117\pi}{5}}$$

Example: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1, y = 0, x = 0, and\ x = 1$ about the y-axis.

$x = \sqrt{y-1}$



WASHER METHOD $\pi \int_1^2 [1]^2 - [\sqrt{y-1}]^2 dy$

DISK METHOD $\pi \int_0^1 [1]^2 dy$

$\pi \int_1^2 2 - y\ dy + \pi \int_0^1 dy$

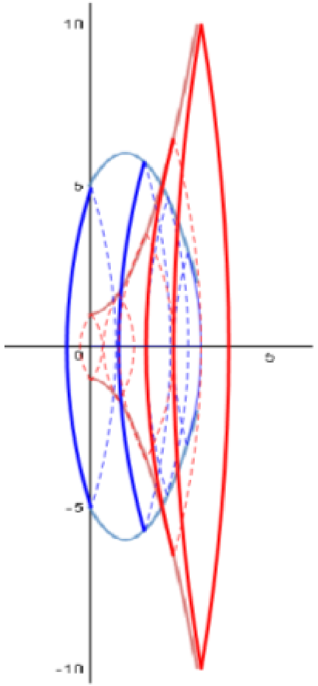
$\pi \left[2y - \frac{1}{2}y^2 \right]_1^2 + \pi \left[y \right]_0^1$

$\pi \left[(4-2) - (2 - \frac{1}{2}) \right] + \pi [1 - 0]$

$\pi (2 - \frac{3}{2}) + \pi$

$\boxed{\frac{3\pi}{2}}$

Example: Find the volume of the solid bounded by the graphs $y = x^2 + 1, y = -x^2 + 2x + 5,$
$x = 0, and\ x = 3$ about the x-axis.



$x^2+1 = -x^2+2x+5$

$2x^2-2x-4=0$

$2(x-2)(x+1)=0$

$x=2,-1$

$$\pi\int_0^2 (-x^2+2x+5)^2-(x^2+1)^2\,dx + \pi\int_2^3 (x^2+1)^2-(-x^2+2x+5)^2\,dx$$

$$\pi\int_0^2 -4x^3-8x^2+20x+24\,dx + \pi\int_2^3 4x^3+8x^2-20x-24\,dx$$

$$\pi\left[-x^4-\frac{8}{3}x^3+10x^2+24x\right]_0^2 + \pi\left[x^4+\frac{8}{3}x^3+10x^2-24x\right]_2^3$$

$$\pi\left[\left(-16-\frac{64}{3}+40+48\right)-(0)\right] + \pi\left[(81+72+90-72)-\left(16+\frac{64}{3}+40-48\right)\right]$$

$$\frac{152}{3}\pi + \frac{425}{3}\pi$$

$$\boxed{\frac{577}{3}\pi}$$

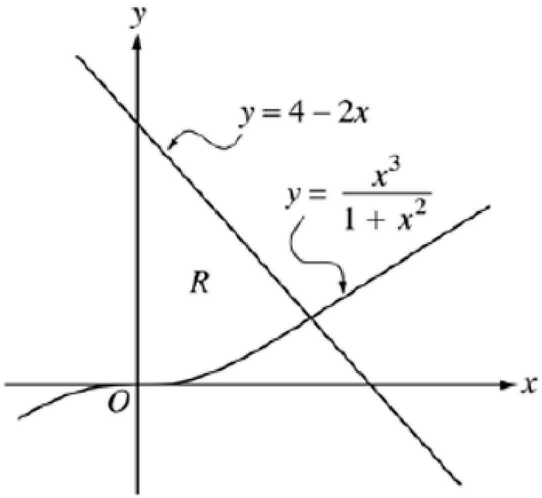
Free Response Practice: 2002 Question 1 (Calculator Active)

Let R be the region bounded by the y-axis and the graphs of $y = \dfrac{x^3}{1+x^2}$ and $y = 4 - 2x$, as shown in the figure above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.



Region R $\frac{x^3}{1+x^2} = 4 - 2x$ at $x = 1.487664 = A$	1 : Correct limits in an integral in (a), (b), or (c).
(a) Area $= \int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right) dx$ $= 3.214$ or 3.215	$2 \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi \int_0^A \left((4-2x)^2 - \left(\frac{x^3}{1+x^2}\right)^2\right) dx$ $= 31.884$ or 31.885 or 10.149π	$3 \begin{cases} 2 : \text{integrand and constant} \\ \quad < -1 > \text{ each error} \\ 1 : \text{answer} \end{cases}$
(c) Volume $= \int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right)^2 dx$ $= 8.997$	$3 \begin{cases} 2 : \text{integrand} \\ \quad < -1 > \text{ each error} \\ \quad \text{note: } 0/2 \text{ if not of the form} \\ \qquad k\int_c^d (f(x) - g(x))^2\, dx \\ 1 : \text{answer} \end{cases}$