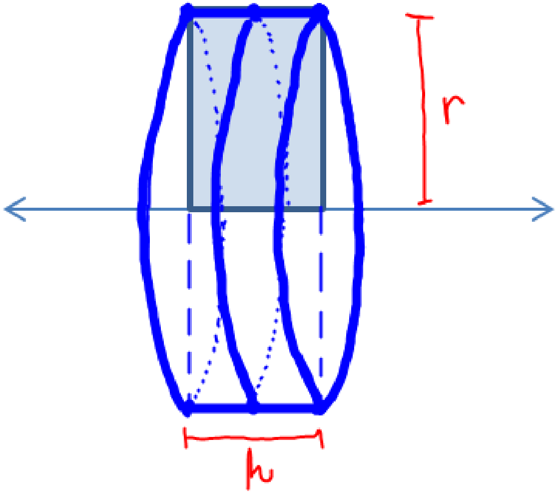
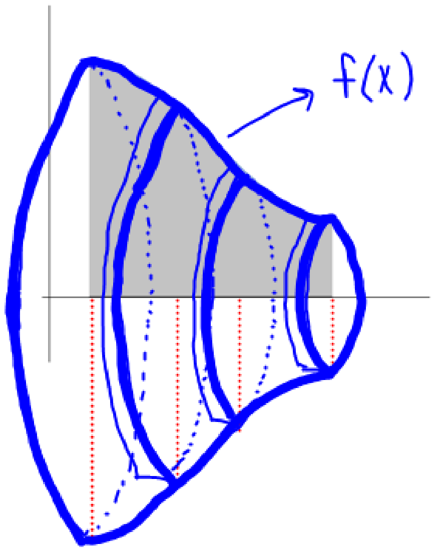
Volumes of Revolution

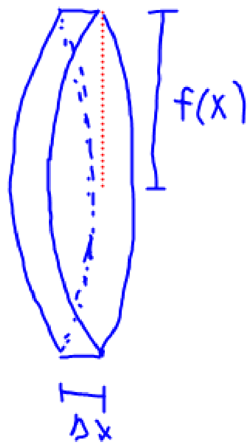
There are several others ways to generate the volume of three dimensional objects in the coordinate plane. Each method involves revolving a region in the plane about a line. This forms a **solid of revolution** and the line is called the **axis of revolution.**

- The simplest such solid is formed by revolving a rectangle about an axis to form a cylinder or **disk.**



$V = \pi r^2 \cdot h$





$V = \pi [f(x)]^2 \cdot \Delta x$

The Disk Method

To find the volume of a solid of revolution with the **disk method**:

Horizontal Axis of Revolution:

$$\pi \int_a^b [f(x)]^2 dx$$

Vertical Axis of Revolution:

$$\pi \int_a^b [f(y)]^2 dy$$

- f(x) is the radius of the disk, which is the function of the curve in question.

Note:

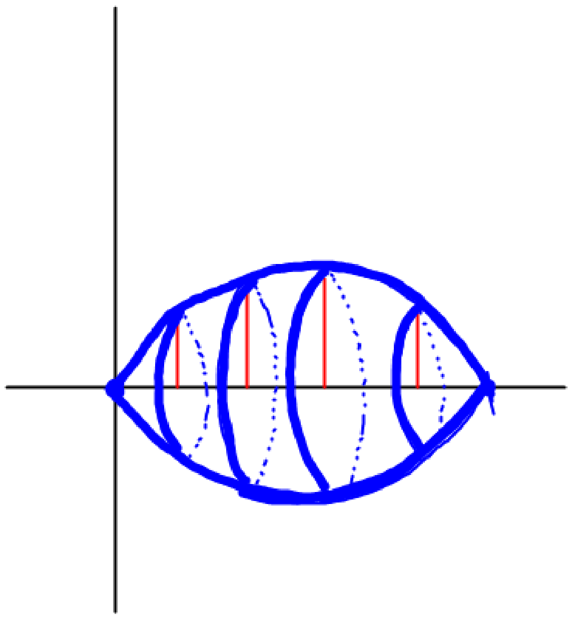
When using the disk method, the representative rectangle is always perpendicular to the axis of revolution.

- If axis of revolution is horizontal, integrate in terms of x.
- If axis of revolution is vertical, integrate in terms of y.

The region in question must be adjacent to the axis of revolution.

Example: Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = x - x^2$ and the x-axis about the x-axis.

$f(x) = x(1-x)$



$$\pi \int_0^1 [x - x^2]^2 \, dx$$

$$\pi \int_0^1 [x^2 - 2x^3 + x^4] \, dx$$

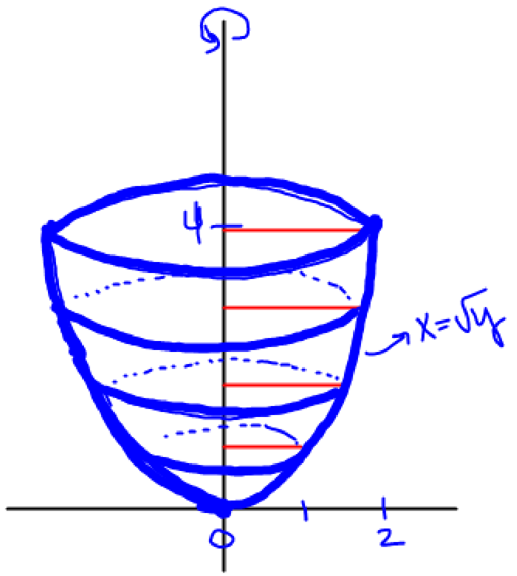
$$\pi \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_0^1$$

$$\pi \left[\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) - (0) \right]$$

$$\boxed{\frac{\pi}{30}}$$

<u>Example:</u> Find the volume of the solid formed by revolving the region bound by the graph of $y = x^2$ on the interval [0, 2] about the y – axis.

$x = \pm\sqrt{y}$



$$\pi \int_0^4 \left[\sqrt{y}\right]^2 dy$$

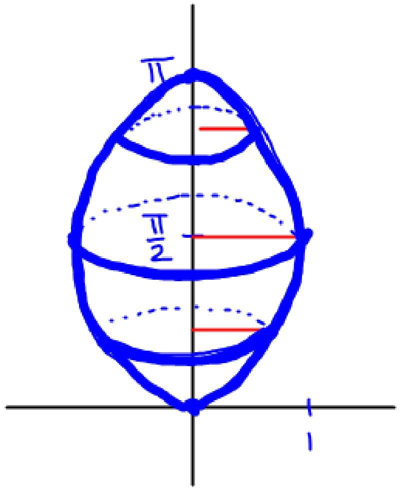
$$\pi \int_0^4 y \, dy$$

$$\pi \left[\tfrac{1}{2}y^2\right]_0^4$$

$$\pi \left[\left(\tfrac{1}{2}(4)^2\right) - \left(\tfrac{1}{2}(0)^2\right)\right]$$

$$\boxed{8\pi}$$

Example: Find the volume of the solid formed by revolving the region bound by the graph of $x = \sqrt{\sin y}$ and the y-axis, on the interval $0 \le y \le \pi$, about the y-axis.



$$\pi \int_0^{\pi} \left[\sqrt{\sin y}\,\right]^2 dy$$

$$\pi \int_0^{\pi} \sin y \; dy$$

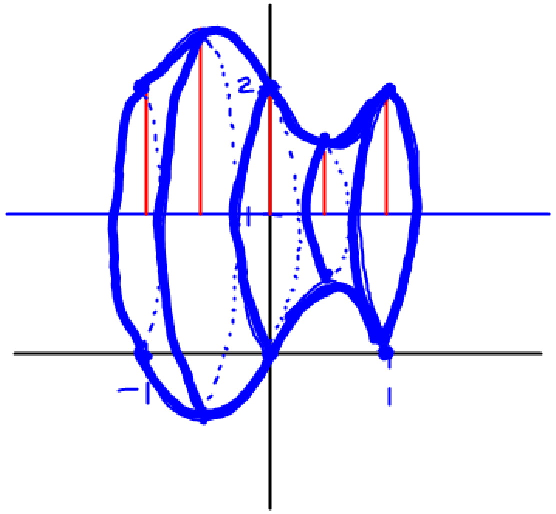
$$\pi \left[-\cos y \right]_0^{\pi}$$

$$\pi \left[-\cos(\pi) + \cos(0) \right]$$

$$\boxed{2\pi}$$

Revolving Around a Different Line

Example: Find the volume of the solid formed by revolving the region bound by the graph of $f(x) = x^3 - x + 2$ and $g(x) = 1$, on the interval [-1, 1], about the line y = 1.



$$\pi \int_{-1}^{1} [x^3 - x + 2 - 1]^2 \, dx$$

$$\pi \int_{-1}^{1} [x^3 - x + 1]^2 \, dx$$

$$\pi \int_{-1}^{1} [x^6 - 2x^4 + 2x^3 + x^2 - 2x + 1] \, dx$$

$$\pi \left[\frac{1}{7}x^7 - \frac{2}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 - x^2 + x \right]_{-1}^{1}$$

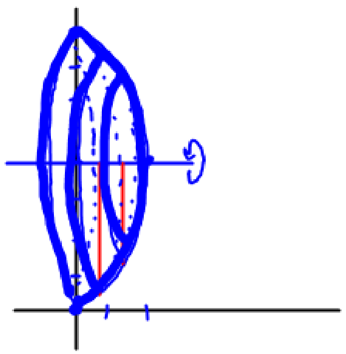
$$\pi \left[\left(\frac{1}{7} - \frac{2}{5} + \frac{1}{2} + \frac{1}{3} - 1 + 1 \right) - \left(-\frac{1}{7} + \frac{2}{5} + \frac{1}{2} - \frac{1}{3} - 1 - 1 \right) \right]$$

$$\pi \left[\left(\frac{121}{210} \right) - \left(-\frac{331}{210} \right) \right]$$

$$\boxed{\frac{226}{105}\pi}$$

Examples: Find the volume of the solid formed by revolving the region bound by the graph of $y = x^2$, on the interval $0 \le x \le 2$, and the given axes.

a. About the line y = 4.

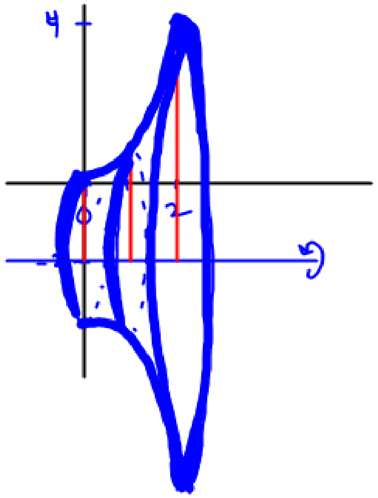


$$\pi \int_0^2 \left[(4) - (x^2)\right]^2 dx$$

$$\pi \int_0^2 \left[16 - 8x^2 + x^4\right] dx$$

$$\pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \pi\left[\left(32 - \frac{64}{3} + \frac{32}{5}\right) - (0)\right] = \boxed{\frac{256}{15}\pi}$$

b. About the line y = -2.



$$\pi \int_0^2 \left[x^2 - (-2)\right]^2 dx$$

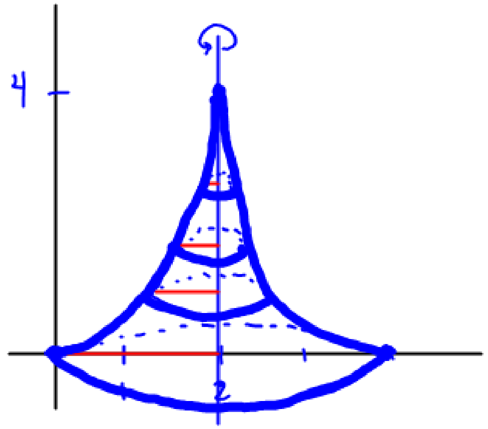
$$\pi \int_0^2 \left[x^4 + 4x^2 + 4\right] dx$$

$$\pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right]_0^2 = \pi\left[\left(\frac{32}{5} + \frac{32}{3} + 8\right) - (0)\right] = \boxed{\frac{376}{15}\pi}$$

Examples: Find the volume of the solid formed by revolving the region bound by the graph of $y = x^2$, on the interval $0 \leq x \leq 2$, and the given axes.

$x = \sqrt{y}$

c. About the line x = 2.

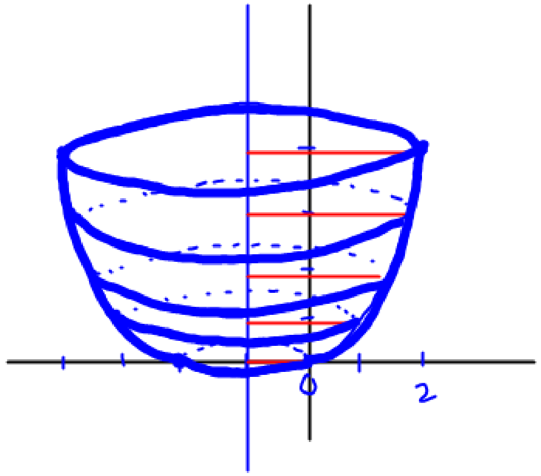


$$\pi \int_0^4 \left[2 - \sqrt{y}\right]^2 dy$$

$$\pi \int_0^4 \left[4 - 4\sqrt{y} + y\right] dy$$

$$\pi \left[4y - \frac{8}{3}y^{3/2} + \frac{1}{2}y^2\right]_0^4 = \pi\left[\left(16 - \frac{64}{3} + 8\right) - (0)\right] = \boxed{\frac{8\pi}{3}}$$

d. About the line x = -1.



$$\pi \int_0^4 \left[\sqrt{y} - (-1)\right]^2 dy$$

$$\pi \int_0^4 \left[y + 2\sqrt{y} + 1\right] dy$$

$$\pi \left[\frac{1}{2}y^2 + \frac{4}{3}y^{3/2} + y\right]_0^4 = \pi\left[\left(8 + \frac{32}{3} + 4\right) - (0)\right] = \boxed{\frac{68}{3}\pi}$$