Volume of Known Cross Sections

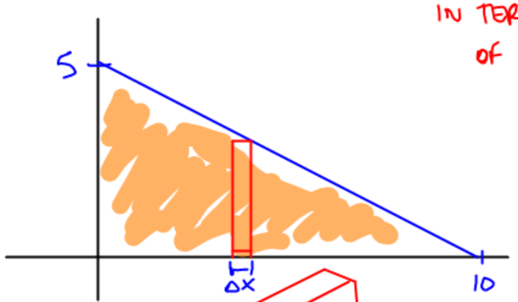
One of the major uses of integrals is to find different areas. Another major use concerns the volume of different figures in the coordinate plane. There a several methods of finding volume depending on the situation.

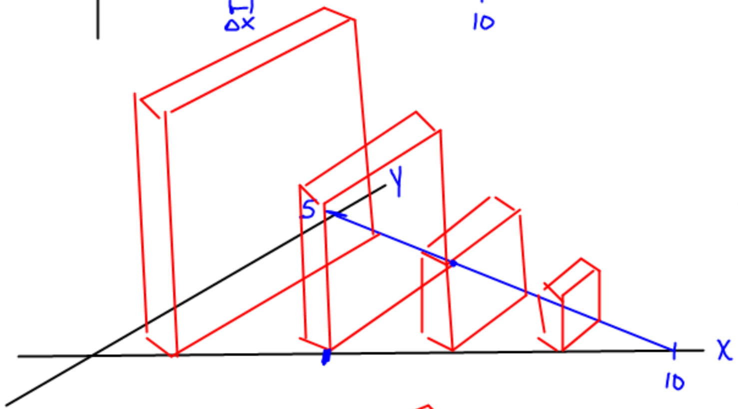
- One method is considering **cross sections** of a 3-dimensional figure. Depending on the shape of the cross sections, the calculation for volume will be set up differently.

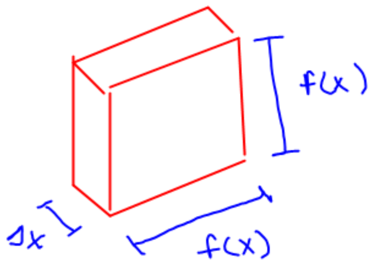
Example: The base of a solid is the region enclosed in the first quadrant by the line $y = \frac{-1}{2}x + 5$. Bases of cross sections are perpendicular to the x-axis. Find the volume of the figures with the following cross sections:

↓ IN TERMS of x

a. Squares.







$V = [f(x)]^2 \cdot \Delta x$

$$\int_0^{10} [f(x)]^2 \, dx$$

$$\int_0^{10} \left[-\tfrac{1}{2}x + 5\right]^2 dx$$

$u = -\frac{1}{2}x + 5$

$du = -\frac{1}{2}dx$

$-2du = dx$

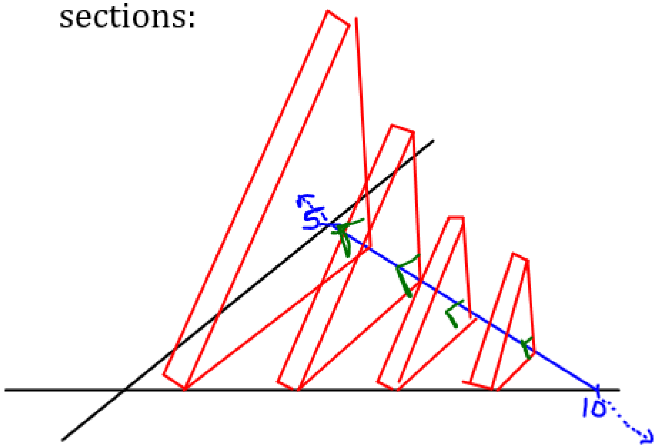
$$-2\int_5^0 u^2 \, du$$

$$2\int_0^5 u^2 \, du$$

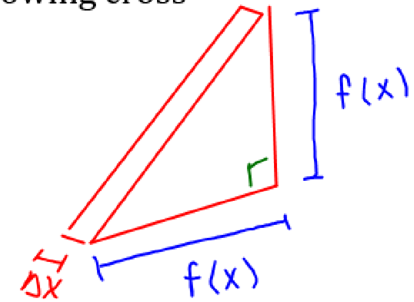
$$\left[\tfrac{2}{3}u^3\right]_0^5$$

$$\boxed{\frac{250}{3}}$$

<u>Example:</u> The base of a solid is the region enclosed in the first quadrant by the line $y = \frac{-1}{2}x + 5$. Bases of cross sections are perpendicular to the x-axis. Find the volume of the figures with the following cross sections:



b. Isosceles Right Triangles.

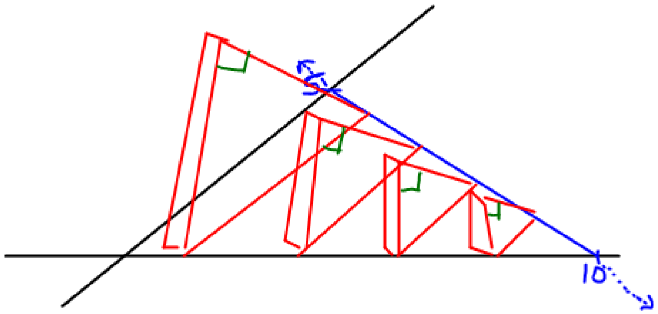


$V = \frac{1}{2}[f(x)]^2 \cdot \Delta x$

<u>LEG IN BASE</u>:

$$\frac{1}{2}\int_0^{10}\left[-\tfrac{1}{2}x+5\right]^2 dx \qquad u = -\tfrac{1}{2}x+5$$

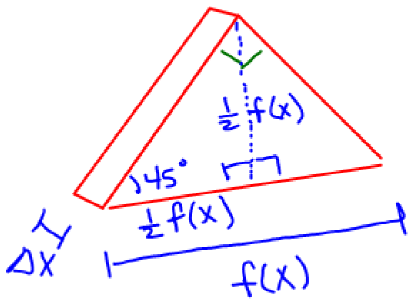
$$\frac{1}{2}\left[2\int_0^5 u^2\,du\right] = \boxed{\frac{125}{3}}$$



<u>HYPOTENUSE IN BASE</u>:

$$\frac{1}{4}\int_0^{10}\left[-\tfrac{1}{2}x+5\right]^2 dx \qquad u = -\tfrac{1}{2}x+5$$

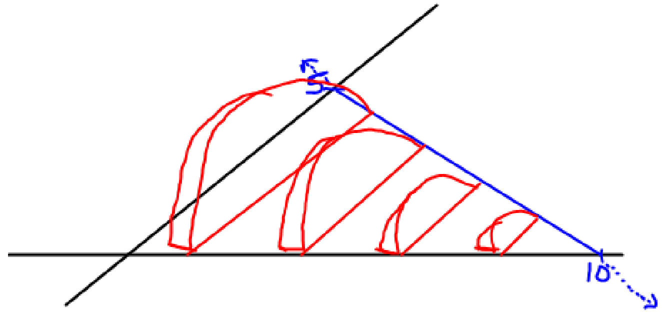
$$\frac{1}{4}\left[2\int_0^5 u^2\,du\right] = \boxed{\frac{125}{6}}$$

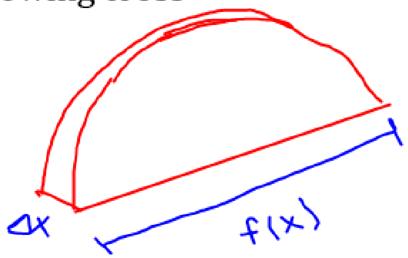


$$V = \frac{1}{2}(f(x))\left(\tfrac{1}{2}f(x)\right)\cdot \Delta x$$
$$= \frac{1}{4}[f(x)]^2 \cdot \Delta x$$

<u>Example:</u> The base of a solid is the region enclosed in the first quadrant by the line $y = \frac{1}{2}x + 5$. Bases of cross sections are perpendicular to the x-axis. Find the volume of the figures with the following cross sections:

c. Semicircles.





DIAMETER IN BASE:

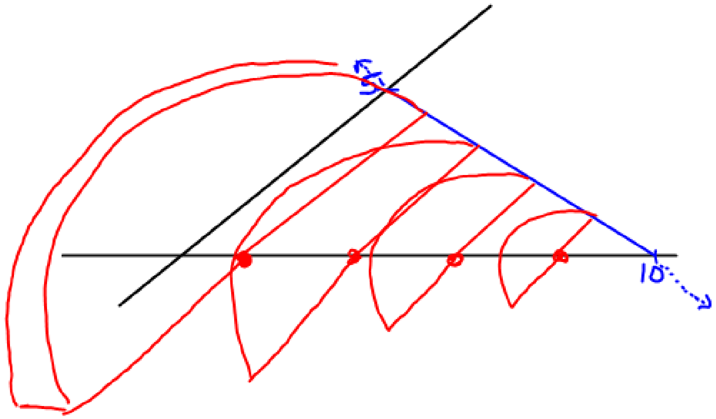
$$V = \frac{1}{2}\left[\pi\left(\tfrac{1}{2}f(x)\right)^2\right]\cdot \Delta x$$

$$V = \frac{\pi}{8}\left[f(x)\right]^2 \cdot \Delta x$$

$$\frac{\pi}{8}\int_0^{10}\left[-\tfrac{1}{2}x+5\right]^2 dx$$

$u = -\frac{1}{2}x+5$

$$\frac{\pi}{8}\left[2\int_0^5 u^2\,du\right] = \boxed{\frac{125\pi}{12}}$$



RADIUS IN BASE:

$$V = \frac{1}{2}\left[\pi\left(f(x)\right)^2\right]\Delta x$$

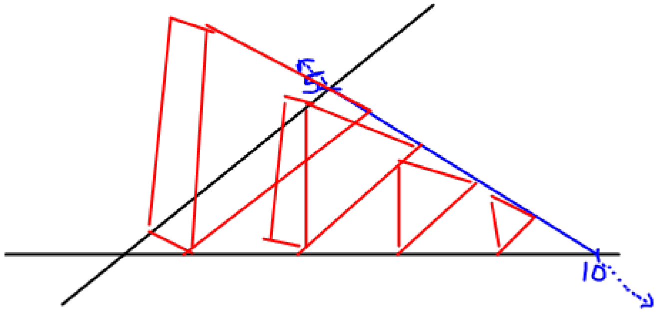
$$\frac{\pi}{2}\int_0^{10}\left[-\tfrac{1}{2}x+5\right]^2 dx$$

$u = -\frac{1}{2}x+5$

$$\frac{\pi}{2}\left[2\int_0^5 u^2\,du\right] = \boxed{\frac{125\pi}{3}}$$

Example: The base of a solid is the region enclosed in the first quadrant by the line $y = -\frac{1}{2}x + 5$. Bases of cross sections are perpendicular to the x-axis. Find the volume of the figures with the following cross sections:

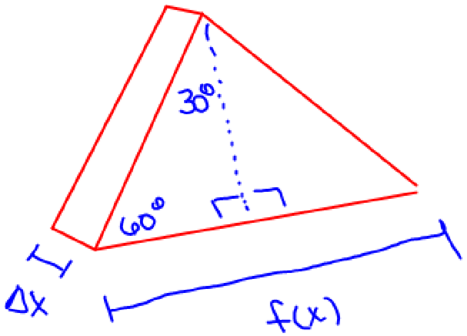
d. Equilateral Triangles.

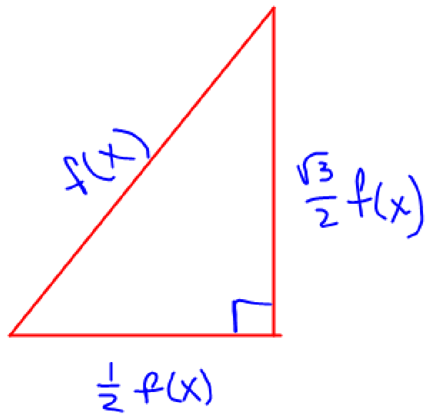


$$\frac{\sqrt{3}}{4}\int_0^{10}\left[-\frac{1}{2}x+5\right]^2 dx$$

$u = -\frac{1}{2}x + 5$

$$\frac{\sqrt{3}}{4}\left[2\int_0^{5} u^2\,du\right] = \boxed{\frac{125\sqrt{3}}{6}}$$





$$V = \frac{1}{2}\cdot f(x)\cdot\frac{\sqrt{3}}{2}f(x)\cdot\Delta x$$
$$= \frac{\sqrt{3}}{4}\left[f(x)\right]^2\cdot\Delta x$$

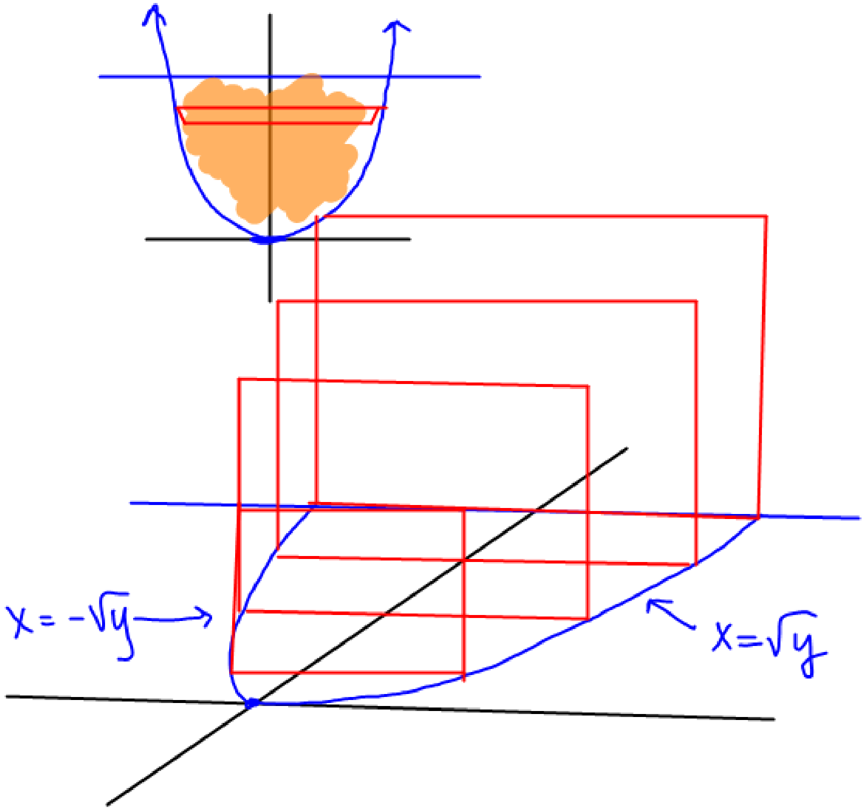
$$\left[\frac{1}{2}f(x)\right]^2 + h^2 = f(x)^2$$
$$\frac{1}{4}f(x)^2 + h^2 = f(x)^2$$
$$\sqrt{h^2} = \sqrt{\frac{3}{4}f(x)^2}$$
$$h = \frac{\sqrt{3}}{2}f(x)$$

Example: The base of the region is bounded by the curves $y = 9$ and $y = x^2$. Bases of cross sections are perpendicular to the y-axis. Find the volume of the figures with the following cross sections:

IN TERMS OF Y

$x = \pm\sqrt{y}$

a. Squares.



$$\int_0^9 \left[\sqrt{y} - -\sqrt{y}\right]^2 dy$$

$$\int_0^9 \left[2\sqrt{y}\right]^2 dy$$

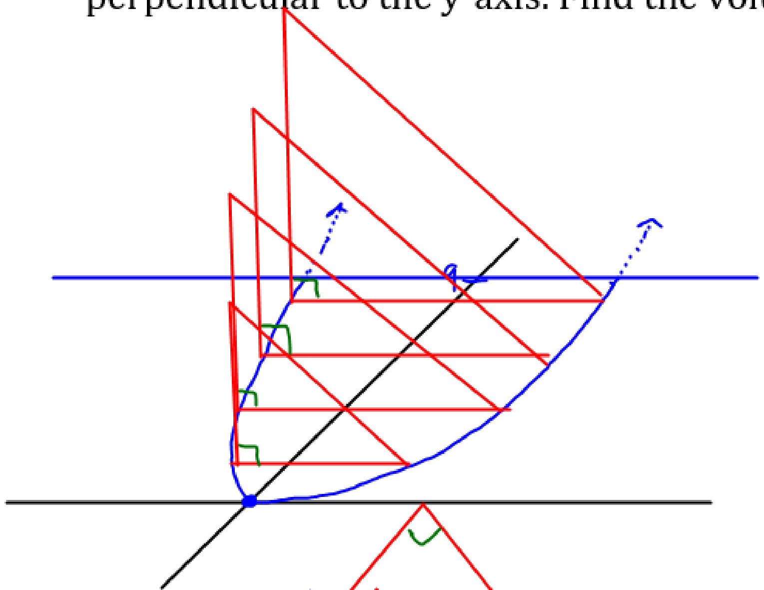
$$4\int_0^9 y \, dy$$

$$4\left[\tfrac{1}{2}y^2\right]_0^9$$

$$2(9)^2 - 2(0)^2 = \boxed{162}$$

<u>Example:</u> The base of the region is bounded by the curves $y = 9$ and $y = x^2$. Bases of cross sections are perpendicular to the y-axis. Find the volume of the figures with the following cross sections:

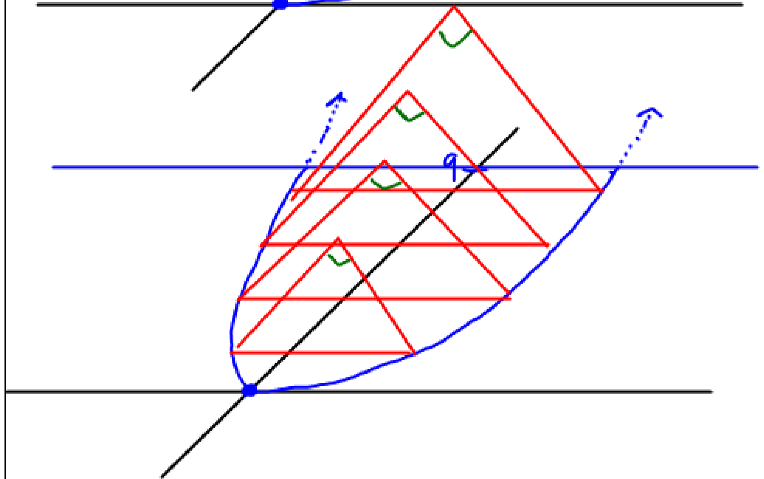
b. Isosceles Right Triangles.



LEG IN BASE

$$\frac{1}{2}\int_0^9 \left[\sqrt{y} - -\sqrt{y}\right]^2 dy$$

$$\frac{1}{2}\left[4\int_0^9 y\,dy\right] = \boxed{81}$$



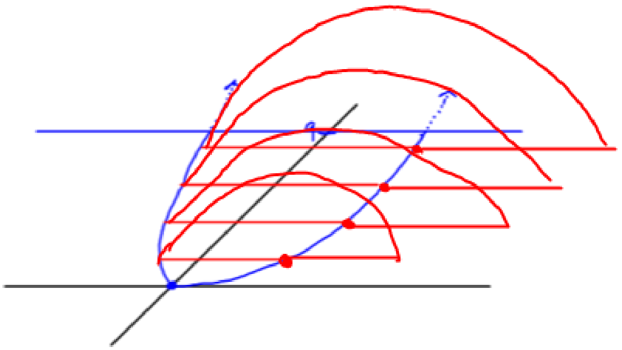
Hyp IN BASE

$$\frac{1}{4}\int_0^9 \left[\sqrt{y} - -\sqrt{y}\right]^2 dy$$

$$\frac{1}{4}\left[4\int_0^9 y\,dy\right] = \boxed{\frac{81}{2}}$$

Example: The base of the region is bounded by the curves $y = 9$ and $y = x^2$. Bases of cross sections are perpendicular to the y-axis. Find the volume of the figures with the following cross sections:

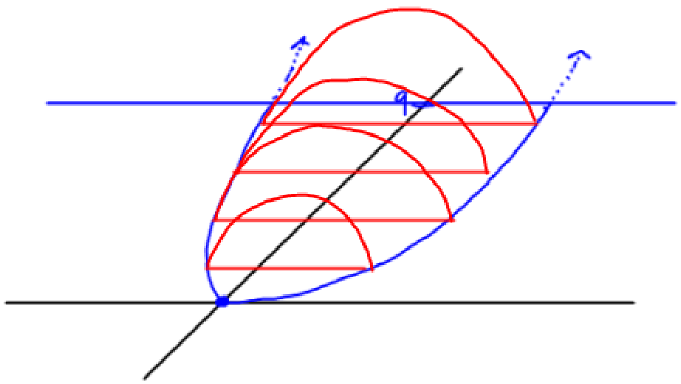
c. Semicircles.



RADIUS :

$$\frac{\pi}{2}\int_0^9 \left[\sqrt{y} - -\sqrt{y}\right]^2 dy$$

$$\frac{\pi}{2}\left[4\int_0^9 y\,dy\right] = \boxed{81\pi}$$

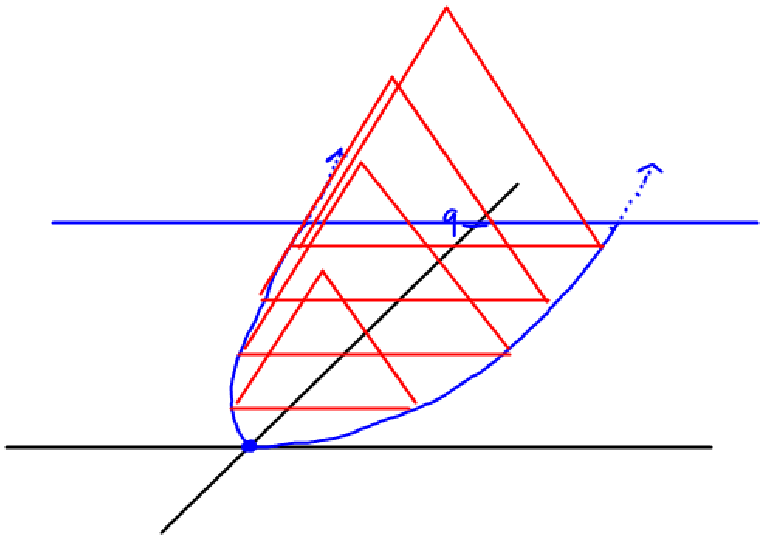


DIAMETER :

$$\frac{\pi}{8}\int_0^9 \left[\sqrt{y} - -\sqrt{y}\right] dy$$

$$\frac{\pi}{8}\left[4\int_0^9 y\,dy\right] = \boxed{\frac{81\pi}{4}}$$

Example: The base of the region is bounded by the curves $y = 9$ and $y = x^2$. Bases of cross sections are perpendicular to the y-axis. Find the volume of the figures with the following cross sections:



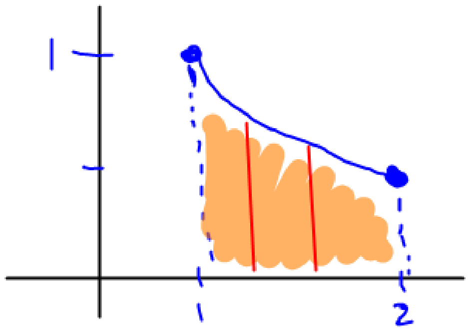
d. Equilateral Triangles.

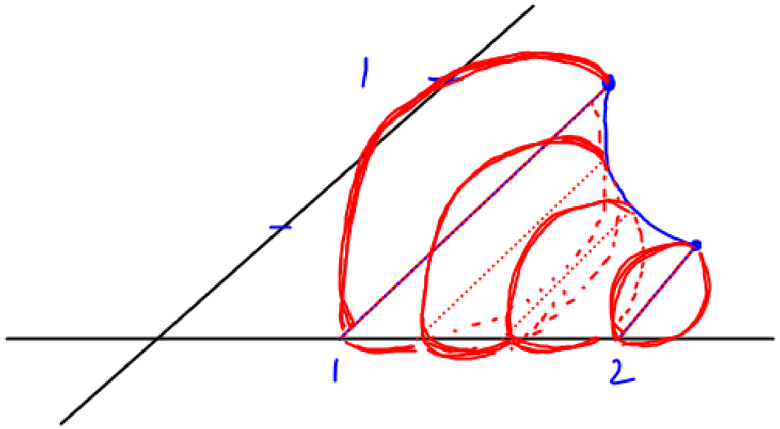
$$\frac{\sqrt{3}}{4}\int_0^9 \left[\sqrt{y} - -\sqrt{y}\right]^2 dy$$

$$\frac{\sqrt{3}}{4}\left[4\int_0^9 y\,dy\right] = \boxed{\frac{81\sqrt{3}}{2}}$$

Volume of Known Cross Sections Continued

Example: The base of a solid is the region between the curve $f(x) = \frac{1}{x}$ and the x-axis on the interval [1, 2]. Cross sections perpendicular to the x axis are circles with diameters in the base.





$$\frac{\pi}{4}\int_1^2 \left[\frac{1}{x}\right]^2 dx$$

$$\frac{\pi}{4}\int_1^2 x^{-2}\, dx$$

$$\frac{\pi}{4}\left[-\frac{1}{x}\right]_1^2$$

$$\frac{\pi}{4}\left[-\frac{1}{2} - -\frac{1}{1}\right] = \frac{\pi}{4}\left[\frac{1}{2}\right] = \boxed{\frac{\pi}{8}}$$

<u>Example:</u> Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the indicated cross sections.

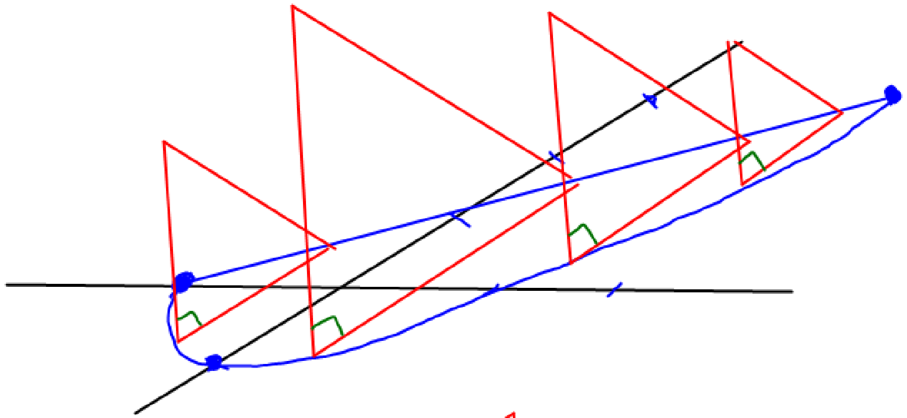
a. Isosceles right triangles, with the hypotenuse in the base, perpendicular to the x-axis.
b. Rectangles perpendicular to the x-axis such that the height is half the base.

$x+1 = x^2 - 1$

$0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$x = -1, 2$



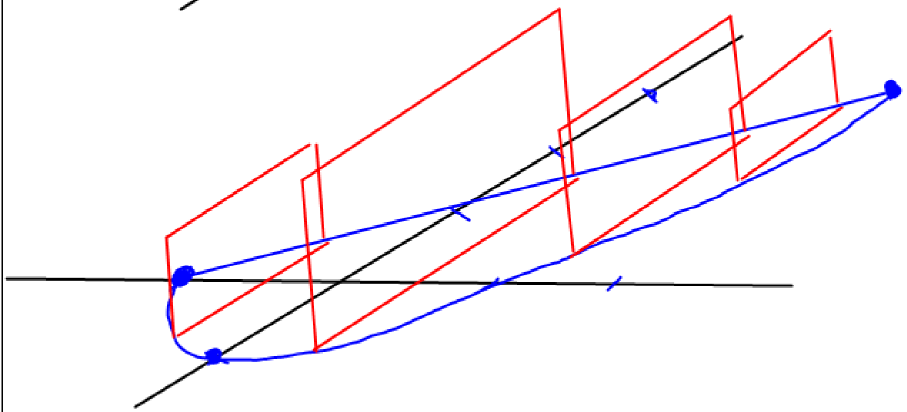
a. $\frac{1}{4}\int_{-1}^{2} \left[(x+1) - (x^2-1)\right]^2 dx$

$\frac{1}{4}\int_{-1}^{2} \left[-x^2+x+2\right]^2 dx$

$\frac{1}{4}\int_{-1}^{2} x^4 - 2x^3 - 3x^2 + 4x + 4$

$\frac{1}{4}\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 - x^3 + 2x^2 + 4x\right]_{-1}^{2}$

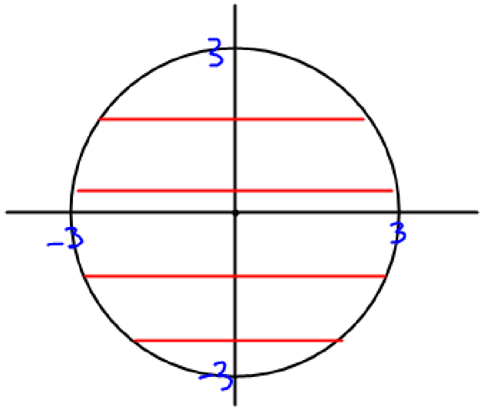
$\frac{1}{4}\left[\left(\frac{32}{5} - 8 - 8 + 8 + 8\right) - \left(-\frac{1}{5} - \frac{1}{2} + 1 + 2 - 4\right)\right] = \boxed{\frac{81}{40}}$

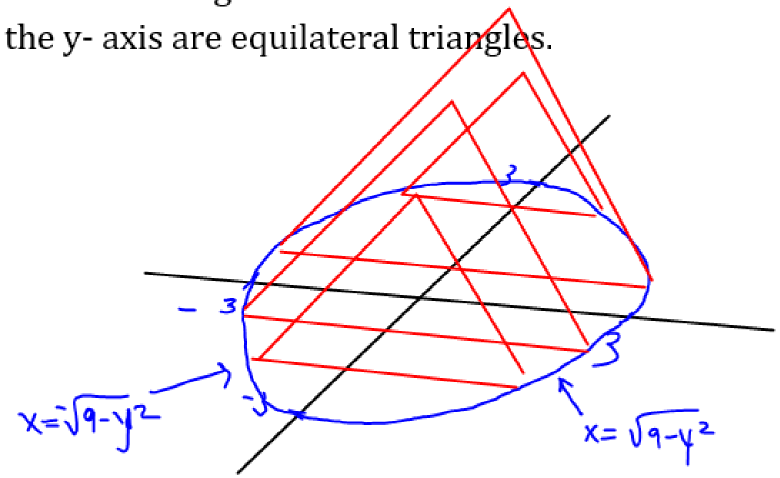


b. $\frac{1}{2}\int_{-1}^{2} \left[(x+1) - (x^2-1)\right]^2 dx$

$\frac{1}{2}\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 - x^3 + 2x^2 + 4x\right]_{-1}^{2} = \boxed{\frac{81}{20}}$

Example: The base of a solid is the circle centered at the origin with radius 3 inches. Find the volume of the solid if all cross sections perpendicular to the y- axis are equilateral triangles.





$x^2+y^2=9$

$x=\pm\sqrt{9-y^2}$

$$\frac{\sqrt{3}}{4}\int_{-3}^{3}\left[\sqrt{9-y^2}--\sqrt{9-y^2}\right]^2 dy$$

$$\frac{\sqrt{3}}{4}\int_{-3}^{3}\left[2\sqrt{9-y^2}\right]^2 dy$$

$$\sqrt{3}\int_{-3}^{3} 9-y^2\,dy$$

$$\sqrt{3}\left[9y-\tfrac{1}{3}y^3\right]_{-3}^{3}$$

$$\sqrt{3}\left[(27-9)-(-27+9)\right]=\boxed{36\sqrt{3}}$$

$x = \ln y$ $\qquad x = \sqrt[3]{y}$

Example: Let R be the region enclosed by the graphs of $y = e^x$ and $y = x^3$, and the y-axis.

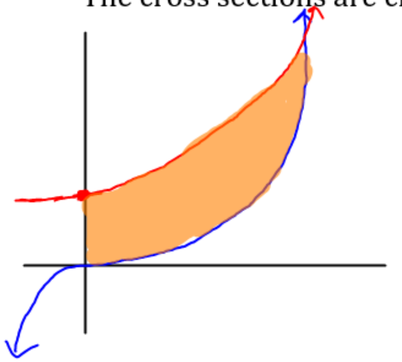
a. Find the volume of the solid with base on region R and cross sections perpendicular to the x-axis. The cross sections are triangles with a height equal to three times the length of the base.

b. Find the volume of the solid with base on region R and cross sections perpendicular to the y-axis. The cross sections are circles whose radius is the length of the base.

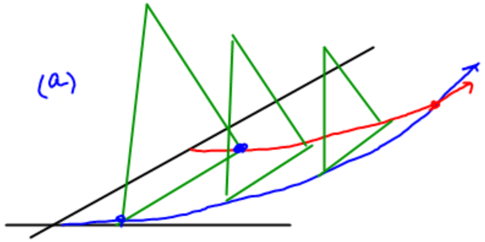
$e^x = x^3$

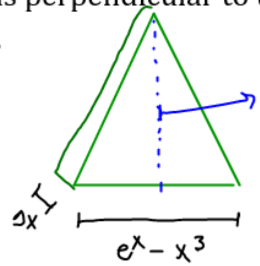
$x = 1.857$

$y = 6.406$



(a)





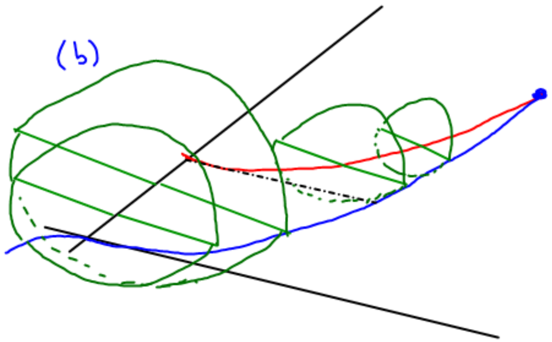
$V = \frac{3}{2}\left[e^x - x^3\right]^2 dx$

$$\frac{3}{2}\int_0^{1.857}\left[e^x - x^3\right]^2 dx = 5.267$$

$$\pi\left[\int_0^1 \left[\sqrt[3]{y}\right]^2 dy + \int_1^{6.406}\left[\sqrt[3]{y} - \ln y\right]^2 dy\right]$$

24.795

(b)



$y = e^x$

$x = \ln y$