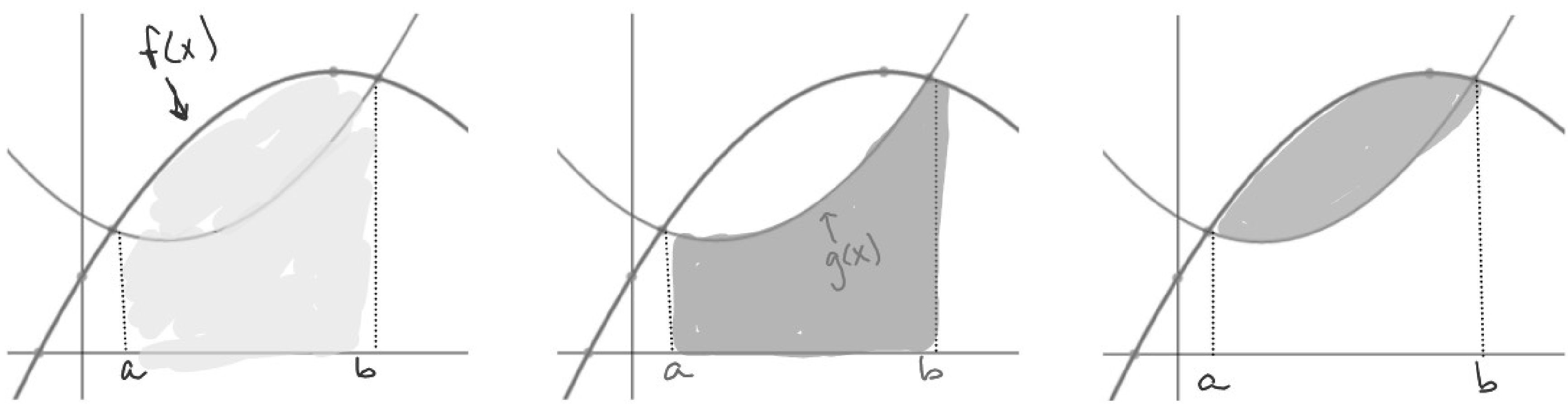
Area Between Curves

Previously we discussed how to use FTC to find the area bounded by the x-axis and the graph of a function. We can extend this to the area of practically any region on the graph, bounded by multiple functions.

Area of a Region Between Two Curves:

Write an integral to represent the area shaded on the following graphs.

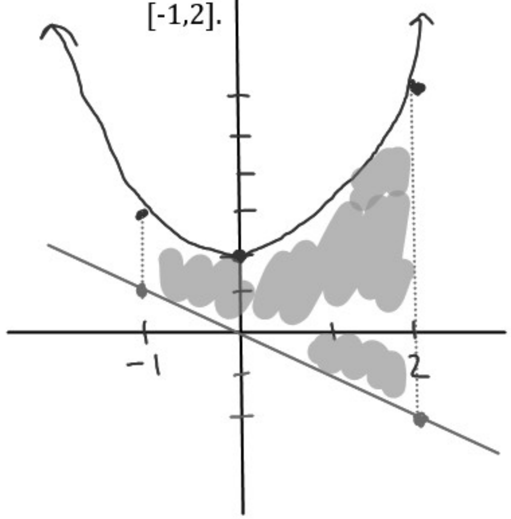


$$\int_a^b f(x)\,dx \quad - \quad \int_a^b g(x)\,dx \quad = \quad \int_a^b \left(f(x)-g(x)\right)\,dx$$

If f and g are continuous on [a,b] and $g(x) \leq f(x)$ for all x in [a,b], then the area of the region bounded by the graphs of f and g on the interval is

$$Area = \int_a^b [\overset{\text{Top}}{f(x)} - \overset{\text{Bottom}}{g(x)}]dx$$

Example: Find the area of the region bounded by the graphs of $y = x^2 + 2$ and $y = -x$ on the interval [-1,2].



$$\int_{-1}^{2} (x^2+2) - (-x) \, dx$$

$$\int_{-1}^{2} x^2 + 2 + x \, dx$$

$$\left(\frac{1}{3}x^3 + 2x + \frac{1}{2}x^2\right)_{-1}^{2}$$

$$\left(\frac{8}{3} + 4 + 2\right) - \left(\frac{-1}{3} - 2 + \frac{1}{2}\right)$$

$$\boxed{\frac{21}{2}}$$

The previous example, the functions do not intersect and you are given the interval to calculate area between. This is not always the case.

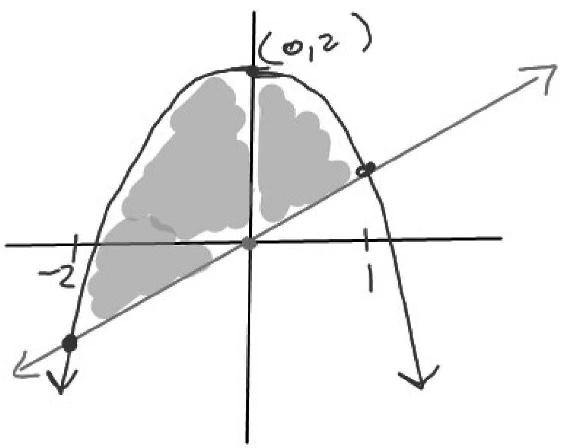
<u>Example:</u> Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

$2 - x^2 = x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$



$$\int_{-2}^{1} (2-x^2) - (x)\,dx$$

$$\int_{-2}^{1} -x^2 - x + 2\,dx$$

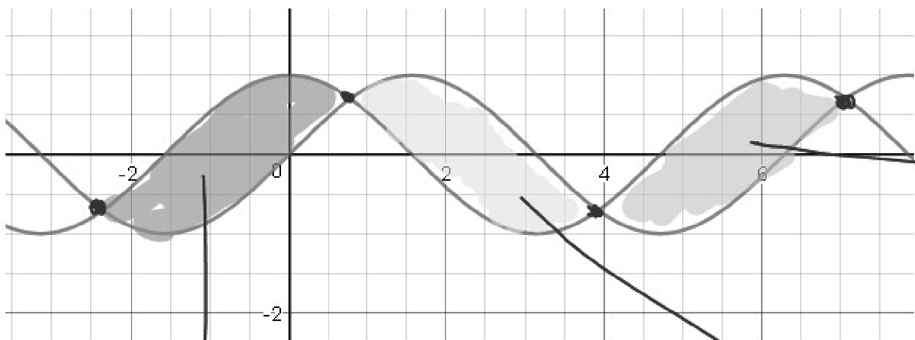
$$\left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1}$$

$$\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)$$

$$\left(\frac{7}{6}\right) - \left(\frac{-10}{3}\right)$$

$$\boxed{\frac{9}{2}}$$

Example: The graphs of $y = \sin x$ and $y = \cos x$ intersect infinitely many times. Find the area of one of the regions bounded by the two graphs.



$\sin x = \cos x$

$x = -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

$\int_{5\pi/4}^{9\pi/4} \cos x - \sin x \, dx$

$\int_{\pi/4}^{5\pi/4} \sin x - \cos x \, dx$

$\int_{-3\pi/4}^{\pi/4} \cos x - \sin x \, dx$

$\left[\sin x + \cos x\right]_{-3\pi/4}^{\pi/4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \boxed{2\sqrt{2}}$

Sometimes, it is very important to graph the functions in order to set up the correct integral.

Example: Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

$= -x(x-2)$

$= x(3x^2 - x - 10)$

$= x(3x+5)(x-2)$

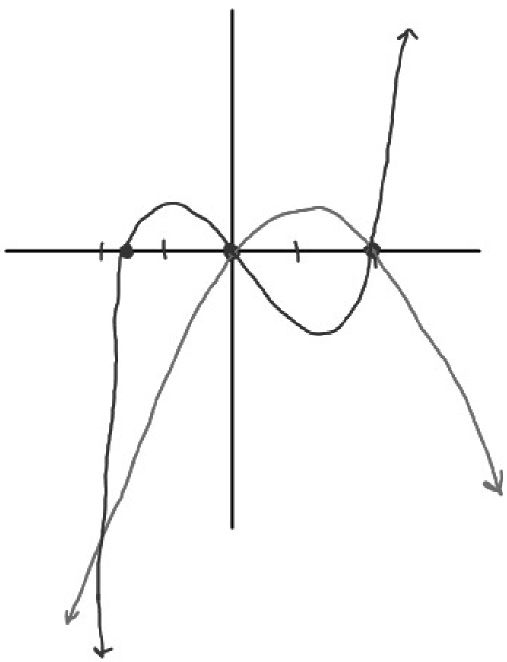
INTERSECTION:

$3x^3 - x^2 - 10x = -x^2 + 2x$

$3x^3 - 12x = 0$

$3x(x^2 - 4) = 0$

$x = 0, \pm 2$



$$\int_{-2}^{0} (3x^3 - x^2 - 10x) - (-x^2 + 2x)\, dx + \int_{0}^{2} (-x^2 + 2x) - (3x^3 - x^2 - 10x)\, dx$$

$$\int_{-2}^{0} 3x^3 - 12x\, dx - \int_{0}^{2} 3x^3 - 12x\, dx$$

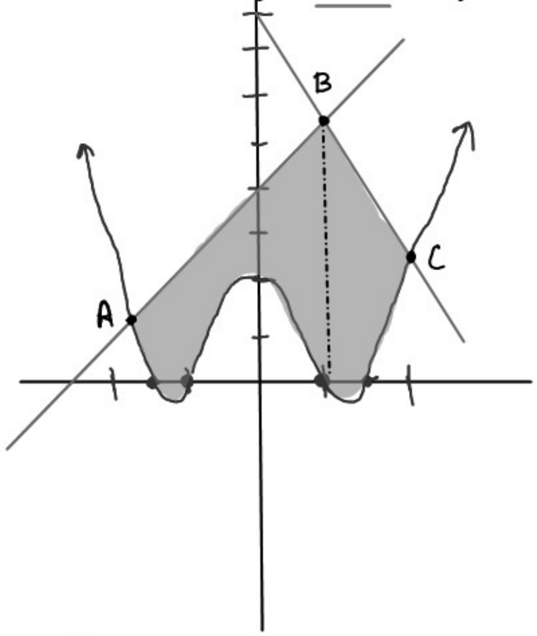
$$\left[\frac{3}{4}x^4 - 6x^2\right]_{-2}^{0} - \left[\frac{3}{4}x^4 - 6x^2\right]_{0}^{2}$$

$$\left[(0) - (12 - 24)\right] - \left[12 - 24\right]$$

$$12 + 12$$

$$\boxed{24}$$

Example: Find the area of the region above the graph of $y = x^4 - 3x^2 + 2$ and below the graphs of $y = x + 4$ and $y = -3x + 8$.



$y = (x^2 - 2)(x^2 - 1)$

$x = \pm\sqrt{2}, \pm 1$

$$\int_{-1.755}^{1} (x+4) - (x^4 - 3x^2 + 2)\,dx + \int_{1}^{1.788} (-3x+8) - (x^4 - 3x^2 + 2)\,dx$$

$$(7.346) + (2.694)$$

$$\boxed{10.040}$$

$\boxed{A}$ $x^4 - 3x^2 - 2 = x + 4 \Rightarrow x = -1.755$

$\boxed{B}$ $x + 4 = -3x + 8 \Rightarrow x = 1$

$\boxed{C}$ $x^4 - 3x^2 - 2 = -3x + 8 \Rightarrow x = 1.788$

Example: Find the area of the region bounded by the graphs of $y = -x^3 - x^2 + 5x,\ y = -x$

$y = -x(x^2 + x - 5) \quad y = -x$

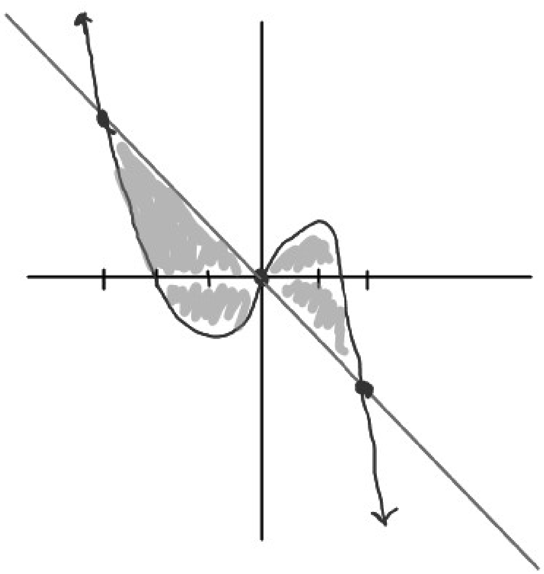
INTERSECTION:

$-x^3 - x^2 + 5x = -x$

$0 = x^3 + x^2 - 6x$

$0 = x(x^2 + x - 6)$

$0 = x(x+3)(x-2)$



$$\int_{-3}^{0} (-x) - (-x^3 - x^2 + 5x)\,dx + \int_{0}^{2} (-x^3 - x^2 + 5x) - (-x)\,dx$$

$$\int_{-3}^{0} x^3 + x^2 - 6x\,dx - \int_{0}^{2} x^3 + x^2 - 6x\,dx$$

$$\left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right]_{-3}^{0} - \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right]_{0}^{2}$$

$$\left[(0) - \left(\frac{81}{4} - 9 - 27\right)\right] - \left[\left(4 + \frac{8}{3} - 12\right) - (0)\right]$$

$$\left(\frac{63}{4}\right) - \left(-\frac{16}{3}\right)$$

$$\boxed{\frac{253}{12}}$$

Example: Find the area of the region in the first quadrant, bounded by the three functions:

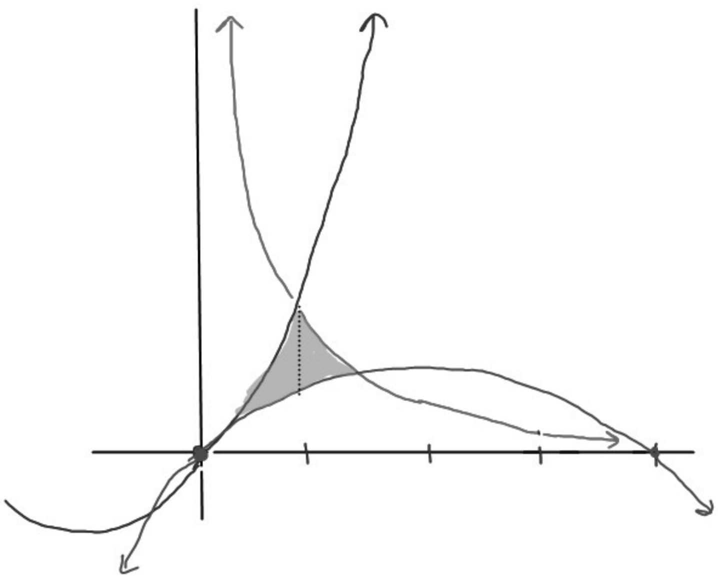
$y = x^2 + 3x$ $\qquad y = \frac{4}{x^2}$ $\qquad y = x - \frac{1}{4}x^2$

$y = x(x+3)$ $\qquad y = \frac{4}{x^2}$ $\qquad y = -\frac{1}{4}x(x-4)$

$x^2+3x = \frac{4}{x^2}$
$x^4 + 3x^3 = 4$
$x = 1$

$x^2+3x = x - \frac{1}{4}x^2$
$4x^2+12x = 4x - x^2$
$5x^2 + 8x = 0$
$x(5x+8) = 0$
$x = 0$

$x - \frac{1}{4}x^2 = \frac{4}{x^2}$
$x^3 - \frac{1}{4}x^4 = 4$
$x = 2$



$$\int_0^1 (x^2+3x) - (x - \tfrac{1}{4}x^2)\,dx + \int_1^2 \left(\tfrac{4}{x^2}\right) - (x - \tfrac{1}{4}x^2)\,dx$$

$$\int_0^1 \tfrac{5}{4}x^2 + 2x\,dx + \int_1^2 4x^{-2} + \tfrac{1}{4}x^2 - x\,dx$$

$$\left[\tfrac{5}{12}x^3 + x^2\right]_0^1 + \left[\tfrac{-4}{x} + \tfrac{1}{12}x^3 - \tfrac{1}{2}x^2\right]_1^2$$

$$\left[\left(\tfrac{5}{12} + 1\right) - (0)\right] + \left[\left(-2 + \tfrac{2}{3} - 2\right) - \left(-4 + \tfrac{1}{12} - \tfrac{1}{2}\right)\right]$$

$$\left[\tfrac{17}{12}\right] + \left[\left(-\tfrac{10}{3}\right) - \left(-\tfrac{53}{12}\right)\right]$$

$$\boxed{\tfrac{5}{2}}$$

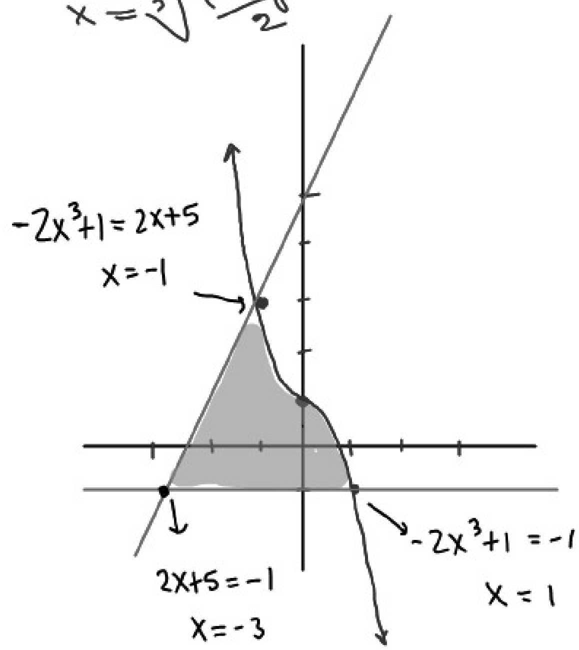
Area Between Curves Continued

Yesterday we discussed how to extend the FTC to find the area between curves. Consider this example:

Example: Find the area of the region bounded on the bottom by $y = -1$ and on top by the curves $y = 2x + 5$ and $y = -2x^3 + 1$.

$x = \frac{y-5}{2}$

$x = \sqrt[3]{\frac{1-y}{2}}$



In Terms x:

$$\int_{-3}^{-1} (2x+5) - (-1)\,dx + \int_{-1}^{1} (-2x^3+1) - (-1)\,dx$$

$$\int_{-3}^{-1} 2x+6\,dx + \int_{-1}^{1} -2x^3+2\,dx$$

$$\left[x^2+6x\right]_{-3}^{-1} + \left[-\tfrac{1}{2}x^4+2x\right]_{-1}^{1} = (1-6)-(9-18)+\left(-\tfrac{1}{2}+2\right)-\left(-\tfrac{1}{2}-2\right)$$

$$= 8$$

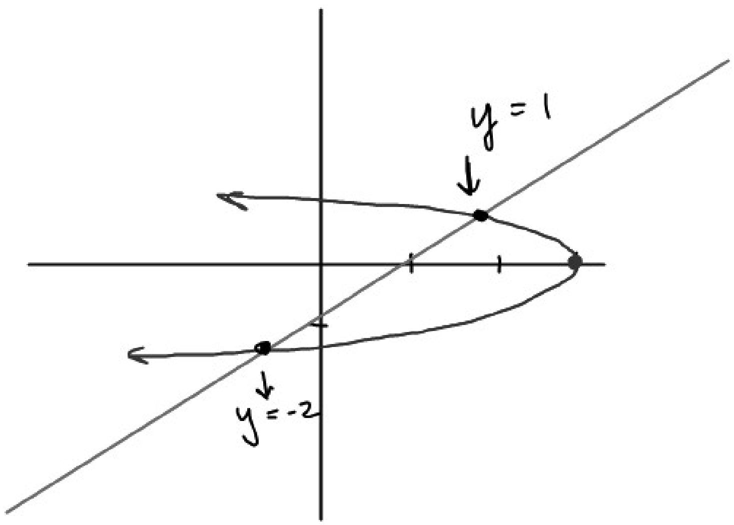
In Terms of y:

$$\int_{-1}^{3} \left(\sqrt[3]{\frac{1-y}{2}}\right) - \left(\frac{y-5}{2}\right)\,dy = 8$$

Being able to integrate in terms of y can be necessary because some relations are defined in terms of y. $y-4=(x-3)^2$

Example: Find the area of the region bound by the graphs of $x = 3 - y^2$ *and* $y = x - 1$.

$x-3=-y^2 \qquad \longrightarrow x = y+1$



$3-y^2 = y+1$

$0 = y^2 + y - 2$

$0 = (y+2)(y-1)$

$$\int_{-2}^{1} (3-y^2)-(y+1)\,dy$$

$$\int_{-2}^{1} -y^2 - y + 2\,dy$$

$$\left[-\tfrac{1}{3}y^3 - \tfrac{1}{2}y^2 + 2y\right]_{-2}^{1}$$

$$\left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) - \left(\tfrac{8}{3} - 2 - 4\right)$$

$$\left(\tfrac{7}{6}\right) - \left(-\tfrac{10}{3}\right)$$

$$\boxed{\tfrac{9}{2}}$$

A Few more examples:

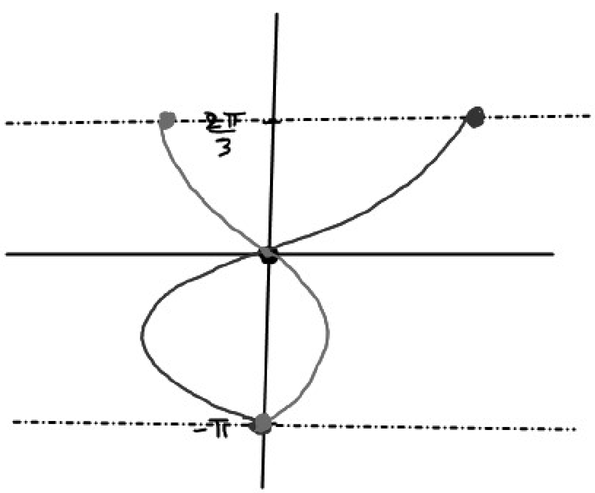
1. Find the area of the regions bounded by the curves $\overset{y=}{\downarrow}$ $= \sin^{-1}(\frac{x}{2})$, $y = \sin^{-1}(-x)$, and the lines $\boxed{y = -\pi, y = \frac{2\pi}{3}}$.

$x = 2\sin y$ $\quad$ $x = -\sin y$

$2\sin y = -\sin y$

$3\sin y = 0$

$y =$



$$\int_{-\pi}^{0} -\sin y - 2\sin y \, dy + \int_{0}^{\frac{2\pi}{3}} 2\sin y - -\sin y \, dy$$

$$-3\int_{-\pi}^{0} \sin y \, dy + 3\int_{0}^{\frac{2\pi}{3}} \sin y \, dy$$

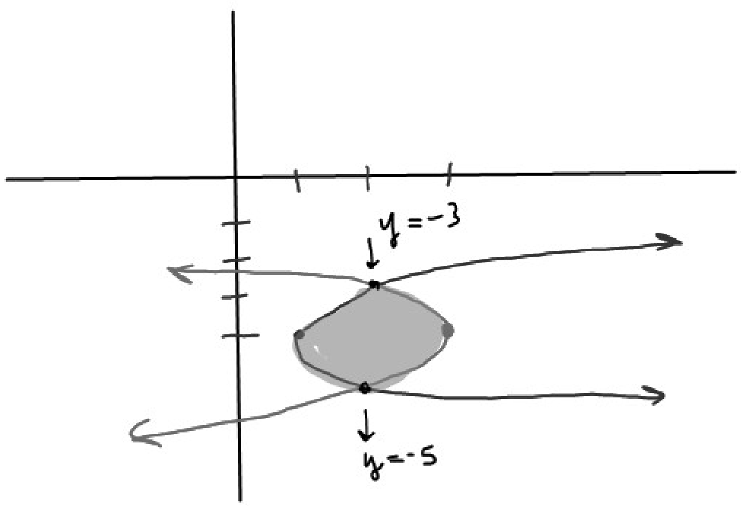
$$3\left[\cos y\right]_{-\pi}^{0} - 3\left[\cos y\right]_{0}^{\frac{2\pi}{3}}$$

$$3\left[(1)-(-1)\right] - 3\left[(-\tfrac{1}{2})-(1)\right]$$

$$\boxed{\frac{21}{2}}$$

2. Find the area bounded by the curves $x = y^2 + 8y + 17$ and $x = -y^2 - 8y - 13$.

$$x - 17 = y^2 + 8y + 16 \qquad x + 13 = -(y^2 + 8y + 16)$$
$$+16 \qquad -16$$
$$x - 1 = (y+4)^2 \qquad x - 3 = -(y+4)^2$$

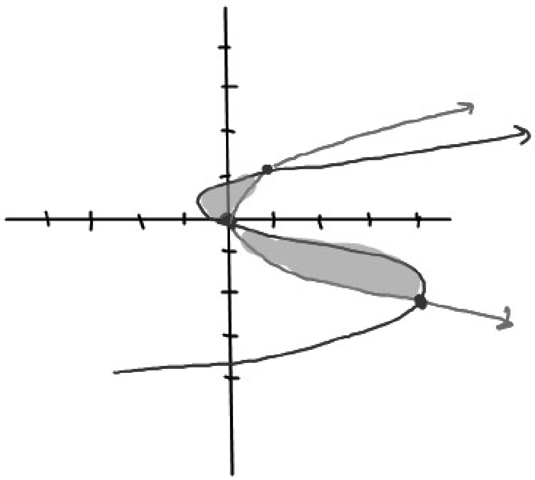


$$y^2 + 8y + 17 = -y^2 - 8y - 13$$
$$2y^2 + 16y + 30 = 0$$
$$y^2 + 8y + 15 = 0$$
$$(y+5)(y+3) = 0$$
$$y = -3, -5$$

$$\int_{-5}^{-3} (-y^2 - 8y - 13) - (y^2 + 8y + 17)\, dy$$
$$-2\int_{-5}^{-3} y^2 + 8y + 15\, dy$$
$$-2\left[\frac{1}{3}y^3 + 4y^2 + 15y\right]_{-5}^{-3}$$
$$-2\left[(-9 + 36 - 45) - \left(\frac{-125}{3} + 100 - 75\right)\right]$$
$$-2\left[(-18) - \left(-\frac{50}{3}\right)\right]$$
$$-2\left[-\frac{4}{3}\right] = \boxed{\frac{8}{3}}$$

3. Find the area bounded by the curves $x = 2y^3 + 3y^2 - 4y$ and $x = y^2$.

$= y(2y^2+3y-4)$ $\quad x=y^2$



INTERSECTION:

$2y^3+3y^2-4y = y^2$

$2y^3+2y^2-4y=0$

$y^3+y^2-2y=0$

$y(y^2+y-2)=0$

$y(y+2)(y-1)=0$

$y=-2, 0, 1$

$x=4 \quad x=0 \quad x=1$

$$\int_{-2}^{0} (2y^3+3y^2-4y)-(y^2)\,dy + \int_0^1 (y^2)-(2y^3+3y^2-4y)\,dy$$

$$\int_{-2}^{0} 2y^3+2y^2-4y\,dy + \int_0^1 -2y^3-2y^2+4y\,dy$$

$$\left[\tfrac{1}{2}y^4+\tfrac{2}{3}y^3-2y^2\right]_{-2}^{0} + \left[-\tfrac{1}{2}y^4-\tfrac{2}{3}y^3+2y^2\right]_0^1$$

$$\left[(0)-\left(8-\tfrac{16}{3}-8\right)\right] + \left[\left(-\tfrac{1}{2}-\tfrac{2}{3}+2\right)-(0)\right]$$

$$\left[\tfrac{16}{3}\right] + \left[\tfrac{5}{6}\right]$$

$$\boxed{\tfrac{37}{6}}$$