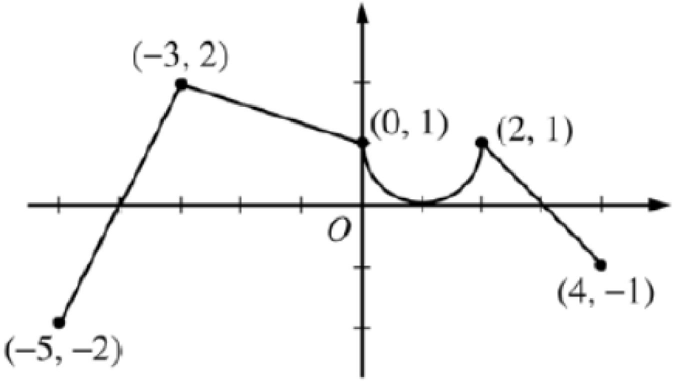

Graph of f

The graph of the function f shown above consists of a semicircle and three line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(0)$ and $g'(0)$.

(b) Find all values of x in the open interval $(-5, 4)$ at which g attains a relative maximum. Justify your answer.

(c) Find the absolute minimum value of g on the closed interval $[-5, 4]$. Justify your answer.

(d) Find all values of x in the open interval $(-5, 4)$ at which the graph of g has a point of inflection.

(a) $g(0) = \int_{-3}^{0} f(t)\,dt = \frac{1}{2}(3)(2+1) = \frac{9}{2}$ $g'(0) = f(0) = 1$	$2 : \begin{cases} 1 : g(0) \\ 1 : g'(0) \end{cases}$
(b) g has a relative maximum at $x = 3$. This is the only x-value where $g' = f$ changes from positive to negative.	$2 : \begin{cases} 1 : x = 3 \\ 1 : \text{justification} \end{cases}$
(c) The only x-value where f changes from negative to positive is $x = -4$. The other candidates for the location of the absolute minimum value are the endpoints. $g(-5) = 0$ $g(-4) = \int_{-3}^{-4} f(t)\,dt = -1$ $g(4) = \frac{9}{2} + \left(2 - \frac{\pi}{2}\right) = \frac{13 - \pi}{2}$ So the absolute minimum value of g is -1.	$3 : \begin{cases} 1 : \text{identifies } x = -4 \text{ as a candidate} \\ 1 : g(-4) = -1 \\ 1 : \text{justification and answer} \end{cases}$
(d) $x = -3, 1, 2$	2 : correct values $\langle -1 \rangle$ each missing or extra value

Particle Motion Revisited

In the past we discussed questions that dealt with the position, velocity and accelerated of a particle. Now, with our study of integrals, we can revisit this concept to discuss some more aspects of particle motion.

Quick Review:

A particle moves along a line such that its position is $s(t) = 2t^3 - 9t^2 + 12t - 4,\ for\ t \geq 0.$

a. Find an equation for the velocity, speed, and acceleration of the particle.
b. When is the distance of the particle increasing? Decreasing?
c. When is the velocity increasing? Decreasing?
d. When is the speed of the particle increasing? Decreasing?

a. $v(t) = s'(t) = 6t^2 - 18t + 12$

SPEED $= |6t^2 - 18t + 12|$

$a(t) = s''(t) = 12t - 18$

b. $v(t) = 0$

$6t^2 - 18t + 12 = 0$

$6(t-2)(t-1) = 0$

$t = 1, 2$ ← AT REST

v(t): + on (0, 1), − on (1, 2) (3/2 marked), + on (2, ∞)

$s(0) = -4$, $s(1) = 1$, $s(2) = 0$

• DIST INC: $(0,1)\ (2,\infty)$

• DIST DEC: $(1,2)$

c. $a(t) = 0$

$12t - 18 = 0$

$t = \frac{3}{2}$

a(t): − on $(0, \frac{3}{2})$, + on $(\frac{3}{2}, \infty)$

v(t) INC: $(\frac{3}{2}, \infty)$

v(t) DEC: $(0, \frac{3}{2})$

d. v(t): + − − +

a(t): − − + +

(number line: 0, 1, $\frac{3}{2}$, 2, ∞)

Speed INC: $(1, \frac{3}{2})\ (2, \infty)$

Speed DEC: $(0, 1)\ (\frac{3}{2}, 2)$

What about integrals?

We already have the relationship that if position is $s(t)$, then:

$$s'(t) = velocity \quad \text{and} \quad s''(t) = acceleration.$$

So it should follow that;

$$\int v(t)dt = position \quad \text{and} \quad \int a(t)dt = velocity$$

Note that these are both indefinite integrals. So in order to find the equation for each you would have to have an initial condition.

Examples:

a. Find the equation for the velocity of a particle if it's acceleration is given by $a(t) = 2t - 6$ and $v(3) = -1$.

$v(t) = \int 2t - 6 \, dt = t^2 - 6t + C$

$3^2 - 6(3) + C = -1$

$C = 8$

$v(t) = t^2 - 6t + 8$

b. Now find the equation for the position of the particle if $s(9) = 62$.

$s(t) = \int t^2 - 6t + 8 \, dt = \frac{1}{3}t^3 - 3t^2 + 8t + C$

$\frac{1}{3} \cdot 9 \cdot 9 \cdot 9 \quad 3 \cdot 9 \cdot 9$

$\frac{1}{3}(9)^3 - 3(9)^2 + 8(9) + C = 62$

$243 - 243 + 72 + C = 62$

$C = -10$

$s(t) = \frac{1}{3}t^3 - 3t^2 + 8t - 10$

We can tell more about a situation when considering the definite integral of velocity (or rate of change) of something.

$\int_a^b v(t)dt$	Represents the **change in position** over the given interval of time. By FTC, the integral will yield $s(b) - s(a)$. This is also known as **displacement.** -If the integral is positive, the particle is ahead of the original position. -If the integral is negative, the particle is behind of the original position.
$\int_a^b \lvert v(t)\rvert dt$	Represents the **total distance traveled** over the given interval of time.

Given $x(p)$ and you want to find $x(q)$ ($x(t)$ is position)

$$x(q) = x(p) + \int_p^q v(t)\,dt$$

$$f(q) = f(p) + \int_p^q f'(t)\,dt$$

Examples:

1. 2017 Question 5 (Non Calculator)

Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$. $\boxed{x_Q(0) = 5}$

(a) For $0 \le t \le 8$, when is particle P moving to the left?

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

(d) Find the position of particle Q the first time it changes direction.

(a) $v_P(t) = \dfrac{1}{t^2-2t+10}\cdot(2t-2)$

$= \dfrac{2(t-1)}{t^2-2t+10}$

cv: $v_P(t) = 0$

$t - 1 = 0$

$t = 1$

$-$ $+$

0   1   8

Particle P moves left $0 < t < 1$.

(b) $v_Q(t) = (t-3)(t-5)$

cv: $v_Q(t) = 0$

$t = 3, 5$

$+$ $-$ $+$

0   3   5   8

$1 < t < 3$ & $5 < t < 8$

(c) $a_Q(t) = 2t - 8$

$a_Q(2) = -4$

Particle Q's speed is decreasing at $t = 2$ since $a_Q(2) < 0$ and $V_Q(2) > 0$.

(since $a_Q(2)$ & $V_Q(2)$ have opposite signs)

FTC

$$\int_0^3 v(t)\,dt = \Big[x(t)\Big]_0^3$$

$$\int_0^3 v(t)\,dt = x(3) - x(0)$$

(d) Q changes direction @ $t = 3$

$$x_Q(3) = x_Q(0) + \int_0^3 V_Q(t)\,dt$$

$$= 5 + \int_0^3 t^2 - 8t + 15\,dt$$

$$= 5 + \left[\tfrac{1}{3}t^3 - 4t^2 + 15t\right]_0^3$$

$$= 5 + \left[\tfrac{1}{3}(27) - 4(9) + 15(3)\right]$$

$$= 5 + [9 - 36 + 45]$$

$$x_Q(3) = 23$$

2. 2016 Question 2 (Calculator)

For $t \geq 0$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right)$. The particle is at position $x = 2$ at time $t = 4$.

(a) At time $t = 4$, is the particle speeding up or slowing down?

(b) Find all times t in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the position of the particle at time $t = 0$.

(d) Find the total distance the particle travels from time $t = 0$ to time $t = 3$.

3. 2013 Question 2 (Calculator)

A particle moves along a straight line. For $0 \le t \le 5$, the velocity of the particle is given by $v(t) = -2 + \left(t^2 + 3t\right)^{6/5} - t^3$, and the position of the particle is given by $s(t)$. It is known that $s(0) = 10$.

(a) Find all values of t in the interval $2 \le t \le 4$ for which the speed of the particle is 2.

(b) Write an expression involving an integral that gives the position $s(t)$. Use this expression to find the position of the particle at time $t = 5$.

(c) Find all times t in the interval $0 \le t \le 5$ at which the particle changes direction. Justify your answer.

(d) Is the speed of the particle increasing or decreasing at time $t = 4$? Give a reason for your answer.

4. 2009 Question 6 (Non Calculator)

t (seconds)	0	8	20	25	32	40
$v(t)$ (meters per second)	3	5	−10	−8	−4	7

The velocity of a particle moving along the x-axis is modeled by a differentiable function v, where the position x is measured in meters, and time t is measured in seconds. Selected values of $v(t)$ are given in the table above. The particle is at position $x = 7$ meters when $t = 0$ seconds.

(a) Estimate the acceleration of the particle at $t = 36$ seconds. Show the computations that lead to your answer. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{20}^{40} v(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate $\int_{20}^{40} v(t)\,dt$.

(c) For $0 \le t \le 40$, must the particle change direction in any of the subintervals indicated by the data in the table? If so, identify the subintervals and explain your reasoning. If not, explain why not.

(d) Suppose that the acceleration of the particle is positive for $0 < t < 8$ seconds. Explain why the position of the particle at $t = 8$ seconds must be greater than $x = 30$ meters.