Integration: Natural Log

Let u be a differentiable function of x.

$$\int \frac{1}{x}dx = \ln|x| + C \qquad \int \frac{1}{u}du = \ln|u| \cdot \frac{dx}{du} + C$$

Examples:

a. $\int \frac{1}{3x+2}dx$

$u = 3x+2$

$du = 3\,dx$

$\frac{1}{3}\int \frac{1}{u}\,du$

$\frac{1}{3}\ln|u| + C$

$\frac{1}{3}\ln|3x+2| + C$

b. $\int \frac{2x}{(x+1)^2}dx$

$u = x+1$

$du = dx$

$x = u-1$

$2\int \frac{u-1}{u^2}\,du$

$2\int \frac{u}{u^2} - \frac{1}{u^2}\,du$

$2\int \frac{1}{u} - u^{-2}\,du$

$2\left[\ln|u| + \frac{1}{u} + C\right]$

$2\left[\ln|x+1| + \frac{1}{x+1}\right] + C$

c. $\int \frac{1}{x \ln x}\,dx$

$$\int \frac{1}{\ln x} \cdot \frac{1}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{1}{u}\,du$$

$$\ln|u| + C$$

$$\boxed{\ln|\ln x| + C}$$

d. $\int \frac{x(x-2)}{(x-1)^3}\,dx$

$u = x - 1$

$du = dx$

$x = u + 1$

$$\int \frac{(u+1)(u-1)}{u^3}\,du$$

$$\int \frac{u^2 - 1}{u^3}\,du$$

$$\int \frac{1}{u} - u^{-3}\,du$$

$$\ln|u| + \frac{1}{2u^2} + C$$

$$\boxed{\ln|x-1| + \frac{1}{2(x-1)^2} + C}$$

e. $\int \frac{1}{1+\sqrt{x}}\,dx$

$u = 1+\sqrt{x}$

$du = \frac{1}{2}x^{-1/2}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$\sqrt{x} = u-1$

$2\,du = \frac{1}{u-1}\,dx$

$2(u-1)\,du = dx$

$\int \frac{1}{u}\cdot 2(u-1)\,du$

$2\int \frac{u-1}{u}\,du$

$2\int 1-\frac{1}{u}\,du$

$2\left[u-\ln|u|\right]+C = 2u-2\ln|u|+C$

$= \boxed{2(1+\sqrt{x})-2\ln|1+\sqrt{x}|+C}$

$= \boxed{2\sqrt{x}-2\ln(1+\sqrt{x})+C}$ → Multiple Choice

Integration with Trig

We have previously discussed the integrals of $\sin x$ and $\cos x$ because they follow directly from differentiation. The integrals of the other trig functions are not so obvious.

Example:

a. $\int \tan x \, dx$

$$\int \frac{\sin x}{\cos x} dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$-\int \frac{1}{u} du$$

$$-\ln|u| + C$$

$$-\ln|\cos x| + C$$

b. $\int \sec x \, dx$

$$\int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) dx$

$$\int \frac{1}{u} du$$

$$\ln|u| + C$$

$$\ln|\sec x + \tan x| + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$