2017 Question 2 (Calculator Active)

When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

(b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

(c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

(d) How many pounds of bananas are on the display table at time $t = 8$?

(a) $\int_0^2 f(t)\,dt = 20.051175$

20.051 pounds of bananas are removed from the display table during the first 2 hours the store is open.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $f'(7) = -8.120$ (or -8.119)

After the store has been open 7 hours, the rate at which bananas are being removed from the display table is decreasing by 8.120 (or 8.119) pounds per hour per hour.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning} \end{cases}$

(c) $g(5) - f(5) = -2.263103 < 0$

Because $g(5) - f(5) < 0$, the number of pounds of bananas on the display table is decreasing at time $t = 5$.

$2 : \begin{cases} 1 : \text{considers } f(5) \text{ and } g(5) \\ 1 : \text{answer with reason} \end{cases}$

(d) $50 + \int_3^8 g(t)\,dt - \int_0^8 f(t)\,dt = 23.347396$

23.347 pounds of bananas are on the display table at time $t = 8$.

$3 : \begin{cases} 2 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

Integration by Substitution

When integrating with composite functions, we can employ a technique called a ***u-substitution.*** The role of the substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall:

$$\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x) \quad \Longrightarrow \quad \int F'(g(x)) \cdot g'(x) = F(g(x)) + c$$

Examples: Find the integral specified.

a. $\int \sqrt{2x-1}\,dx$

$\int (2x-1)^{1/2}\,dx$

$\frac{d}{dx}[u = 2x-1]$

$\cancel{dx} \cdot \frac{du}{\cancel{dx}} = 2 \cdot dx$

$du = 2dx$

$\frac{1}{2} du = dx$

$\int u^{1/2} \cdot \frac{1}{2} du$

$\frac{1}{2} \int u^{1/2} du$

$\frac{1}{2}\left[\frac{2}{3} u^{3/2} + C\right]$

$\frac{1}{3}(2x-1)^{3/2} + C$

$\cancel{\frac{3}{2}} \cdot \frac{1}{\cancel{3}} \boxed{(2x-1)^{1/2}} \cdot \cancel{2}$

b. $\int 2x(x^2+1)^2\,dx$

$\boxed{\int (x^2+1)^2 \cdot 2x\,dx}$

$\frac{d}{dx}[u = x^2+1]$

$\frac{du}{dx} = 2x$

$du = 2x\,dx \longrightarrow \frac{du}{2x} = dx$

$\int u^2 \cdot \cancel{2x} \cdot \frac{du}{\cancel{2x}}$

$\int u^2 du$

$\int u^2 du$

$\frac{u^3}{3} + C$

$\boxed{\frac{(x^2+1)^3}{3} + C}$

c. $\int \sin^2 3x \cos 3x \; dx$

$\int (\sin(3x))^2 \cdot \cos(3x)\,dx$

$u = \sin(3x)$

$du = \cos(3x) \cdot 3\,dx$

$\frac{1}{3}\,du = \cos(3x)\,dx$

$\frac{1}{3}\int u^2\,du$

$\frac{1}{3}\left[\frac{u^3}{3} + C\right]$

$\boxed{\frac{1}{9}\sin^3(3x) + C}$

d. $\int x\sqrt{2x-1}\,dx$

$u = 2x - 1 \qquad x = \frac{u+1}{2}$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

$\frac{1}{2}\int x \cdot u^{1/2}\,du$

$\frac{1}{2}\int \left(\frac{u+1}{2}\right) \cdot u^{1/2}\,du$

$\frac{1}{4}\int u^{3/2} + u^{1/2}\;du$

$\frac{1}{4}\left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C\right]$

$\boxed{\frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C}$

Definite Integrals

When working with definite integrals, there are two ways to approach evaluating the integrals.

1) After integration, rewrite in terms of x and use the original limits of integration.

Examples:

a. $\displaystyle\int_{-1=x}^{1=x} \frac{x+1}{(x^2+2x+2)^3}\,dx$

$u = x^2+2x+2$

$du = (2x+2)dx$

$du = 2(x+1)dx$

$\frac{1}{2}du = (x+1)dx$

$\frac{1}{2}\displaystyle\int \frac{1}{u^3}\,du$

$\frac{1}{2}\displaystyle\int u^{-3}\,du$

$\frac{1}{2}\left[\frac{u^{-2}}{-2}\right]$

$\left[-\frac{1}{4(x^2+2x+2)^2}\right]_{-1}^{1}$

$\frac{-1}{4(25)} - \frac{-1}{4(1)^2} = \boxed{\frac{6}{25}}$

b. $\displaystyle\int_{0=x}^{\pi^2=x} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

$u = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}dx$

$\displaystyle\int_0^{\pi^2} \sin(x^{1/2})\cdot x^{-1/2}\,dx$

$2\displaystyle\int \sin(u)\,du$

$\left[-2\cos(u)\right]$

$\left[-2\cos(\sqrt{x})\right]_0^{\pi^2}$

$-2\cos(\pi) - -2\cos(0) = \boxed{4}$

2) Redefine the limits of integration in terms of u.

Examples:

a. $\int_{0\,=\,x}^{1\,=\,x} x(x^2+1)^3\,dx$

$u = x^2 + 1$

$du = 2x\,dx$

$x=0,\ u=1$

$x=1,\ u=2$

$\frac{1}{2}\int_1^2 u^3\,du$

$\left[\frac{1}{8}u^4\right]_1^2$

$\frac{1}{8}(2)^4 - \frac{1}{8}(1)^4 = \boxed{\frac{15}{8}}$

b. $\int_0^{\pi} \cos x\sqrt{\sin x}\,dx$

$u = \sin x$

$du = \cos x\,dx$

$x=0,\ u=\sin(0)=0$

$x=\pi,\ u=\sin(\pi)=0$

$\int_0^0 \sqrt{u}\,du$

$\boxed{0}$