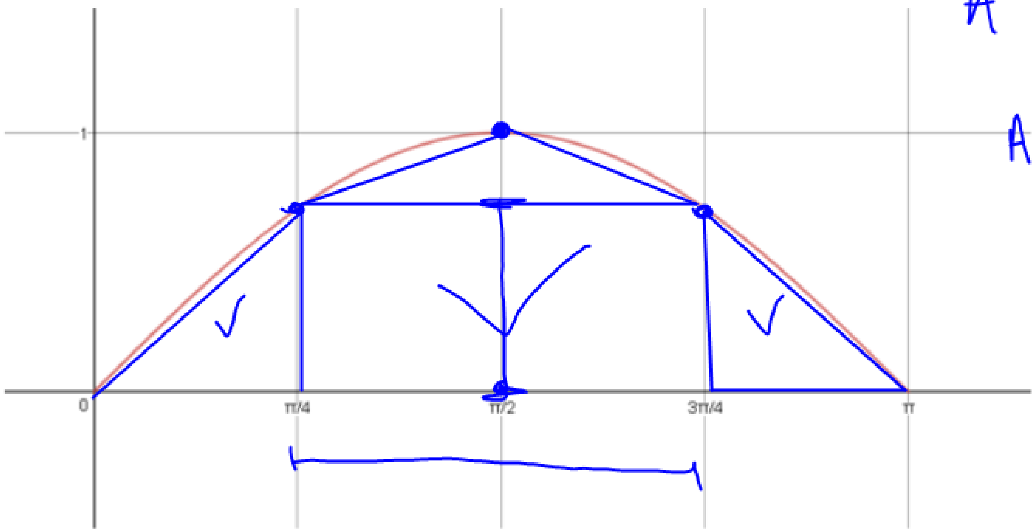
Calculus

Area under a Curve

-The second biggest topic covered in calculus is the idea of finding the area that bounded between a graph of a function and the x-axis. The importance of being able to calculate this area accurately is fundamental to calculus.

-How would you estimate the area between the x-axis and the graph of $y = \sin x$ on the interval from $[0,\pi]$?



$$A \approx \tfrac{1}{2}\pi(1) = \frac{\pi}{2} \approx 1.571$$

$$A \approx \cancel{2}\left[\cancel{\tfrac{1}{2}}\left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right)\right] + \left(\frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)$$

$$\approx \cancel{\frac{\pi\sqrt{2}}{8}} + \frac{\pi\sqrt{2}}{4} + \frac{\pi}{4} - \cancel{\frac{\pi\sqrt{2}}{8}}$$

$$\approx 1.896$$

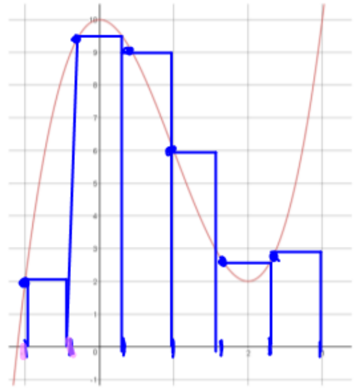
-One way to approximate area under a curve is to use what is called a **Riemann Sum**. When calculating using a Riemann Sum over a given interval, the interval is divided into a specified number of **subintervals.** There are several ways to calculate Riemann Sums depending on how you approach the problem.

$\boxed{\text{Left Endpoint}}$ Riemann Sum

Example 1: Using a left endpoint Riemann sum with 4 subintervals of equal length and 6 subintervals of equal length, approximate the area under the curve of the function $y = 2x^3 - 6x^2 + 10$ on the interval [-1, 3].

4 SUBS $\Delta x = \frac{3 - -1}{4} = 1$

6 SUBS $\Delta x = \frac{3 - -1}{6} = \frac{2}{3}$



$\text{AREA} \approx \Delta x \cdot f(-1) + \Delta x \cdot f(0) + \Delta x \cdot f(1) + \Delta x \cdot f(2)$

$\approx 1 \cdot [2] + 1 \cdot [10] + 1 [6] + 1 \cdot [2]$

≈ 20

$\text{AREA} \approx \Delta x\, f(-1) + \Delta x\, f(-\tfrac{1}{3}) + \Delta x\, f(\tfrac{1}{3}) + \Delta x\, f(1) + \Delta x f(\tfrac{5}{3}) + \Delta x\, f(\tfrac{7}{3})$

$\approx \frac{2}{3}\left[2 + 9.259 + 9.407 + 6 + 2.593 + 2.741\right]$

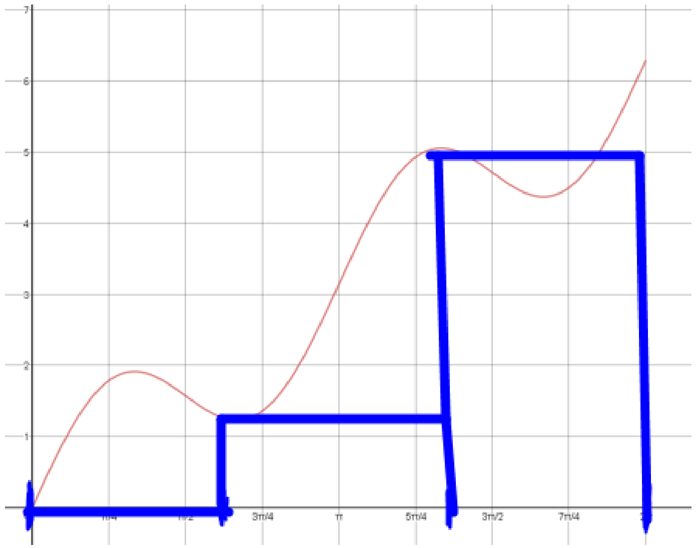
$\approx \frac{2}{3}[32]$

$\approx \frac{64}{3} = 21.333$

<u>Example 2:</u> Using a left endpoint Riemann sum with 3 subintervals of equal length and 4 subintervals of equal length, approximate the area under the curve of the function $y = \sin(2x) + x$ on the interval $[0, 2\pi]$.

3 SUBS

$\Delta x = \frac{2\pi - 0}{3} = \frac{2\pi}{3}$



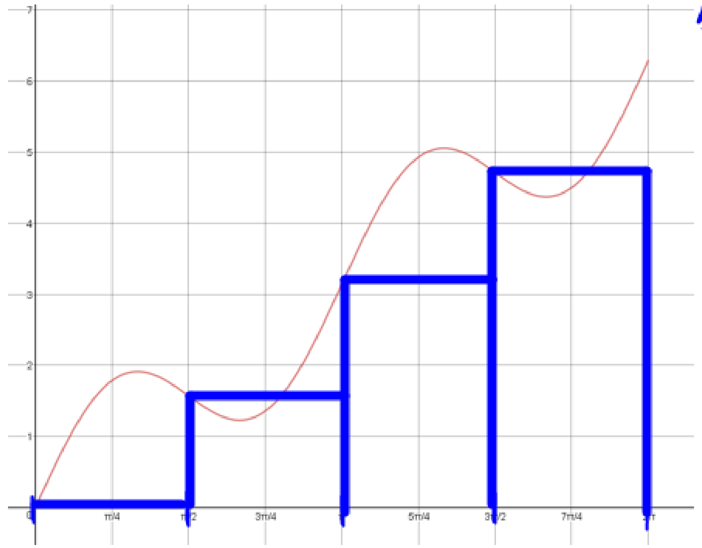
$\text{AREA} \approx \frac{2\pi}{3}\left[y(0) + y\left(\frac{2\pi}{3}\right) + y\left(\frac{4\pi}{3}\right)\right]$

$\approx \frac{2\pi}{3}[0 + 1.228 + 5.055]$

$\approx \frac{2\pi}{3}[6.283] = 13.159$

4 SUBS

$\Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}$



$\text{Area} \approx \frac{\pi}{2}\left[y(0) + y\left(\frac{\pi}{2}\right) + y(\pi) + y\left(\frac{3\pi}{2}\right)\right]$

$\approx \frac{\pi}{2}[0 + 1.571 + 3.142 + 4.712]$

$\approx \frac{\pi}{2}[9.425] = 14.805$

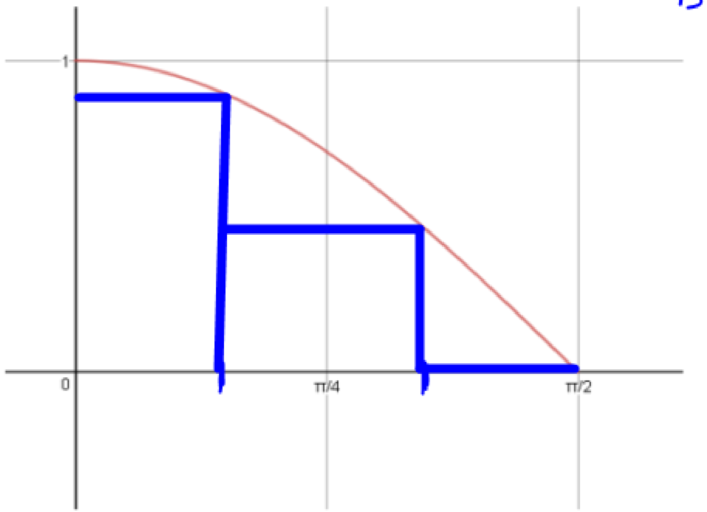
Right Endpoint Riemann Sum

Example 3: Using a right endpoint Riemann sum with 3 subintervals of equal length and 4 subintervals of equal length, approximate the area under the curve of the function $\boxed{y = \cos x}$ on the interval $[0, \frac{\pi}{2}]$.

3 subs

$$\Delta x = \frac{\frac{\pi}{2} - 0}{3}$$

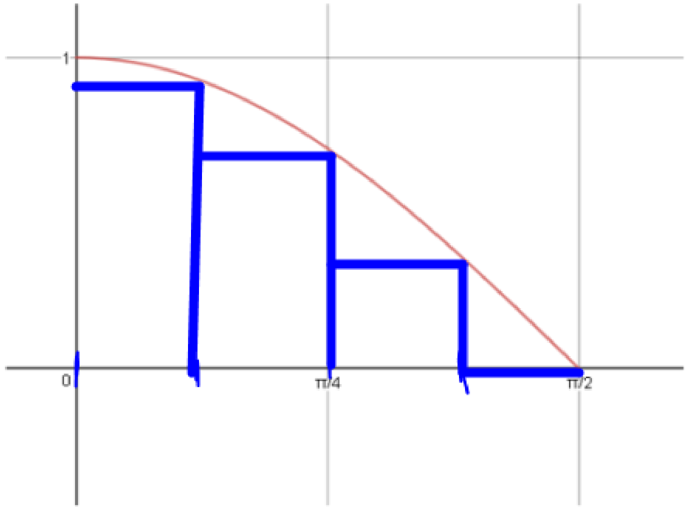
$$\Delta x = \frac{\pi}{6}$$



4 subs

$$\Delta x = \frac{\frac{\pi}{2} - 0}{4}$$

$$\Delta x = \frac{\pi}{8}$$



$$\text{Area} \approx \frac{\pi}{6}\left[f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right)\right]$$

$$\approx \frac{\pi}{6}\left[\frac{\sqrt{3}}{2} + \frac{1}{2} + 0\right]$$

$$\approx \frac{\pi(\sqrt{3}+1)}{12}$$

$$\text{Area} \approx \frac{\pi}{8}\left[f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right]$$

$$\approx \frac{\pi}{8}\left[0.924 + 0.707 + 0.383 + 0\right]$$

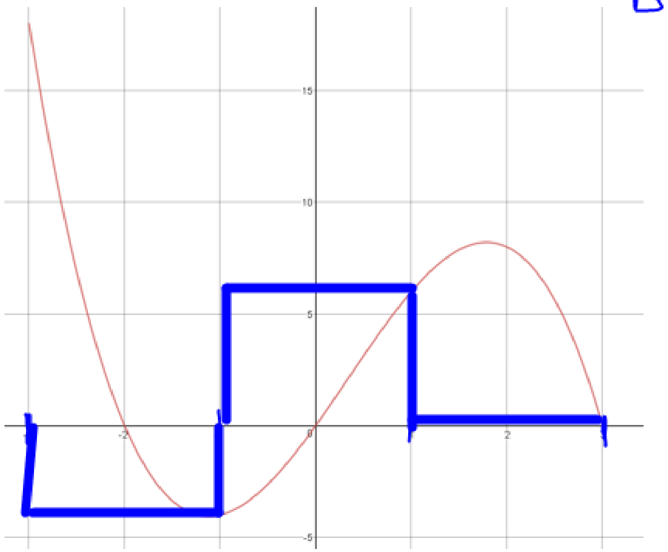
$$\approx \frac{\pi}{8}\left[2.014\right] = 0.791$$

<u>Example 4:</u> Using a right endpoint Riemann sum with 3 subintervals of equal length and 6 subintervals of equal length, approximate the area under the curve of the function $\boxed{y = -x^3 + x^2 + 6x}$ on the interval [-3, 3].

3 SUBS

$\Delta x = \frac{3 - -3}{3}$

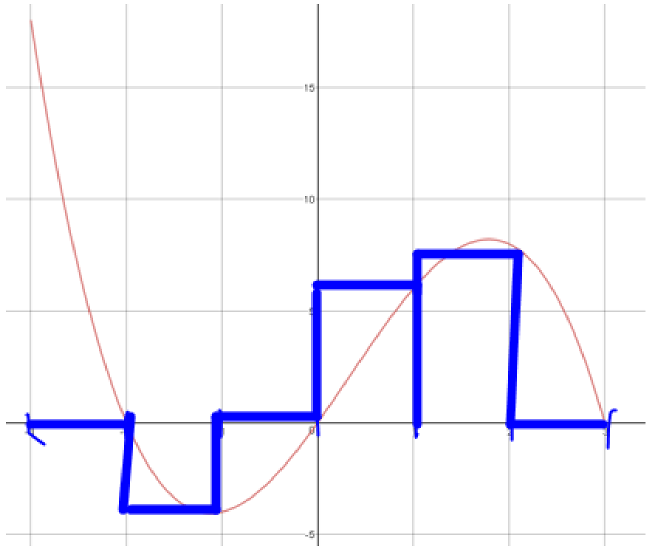
$\Delta x = 2$



6 SUBS

$\Delta x = \frac{3 - -3}{6}$

$= 1$



$\text{AREA} \approx 2\left[f(-1) + f(1) + f(3) \right]$

$\approx 2\left[-4 + 6 + 0 \right]$

≈ 4

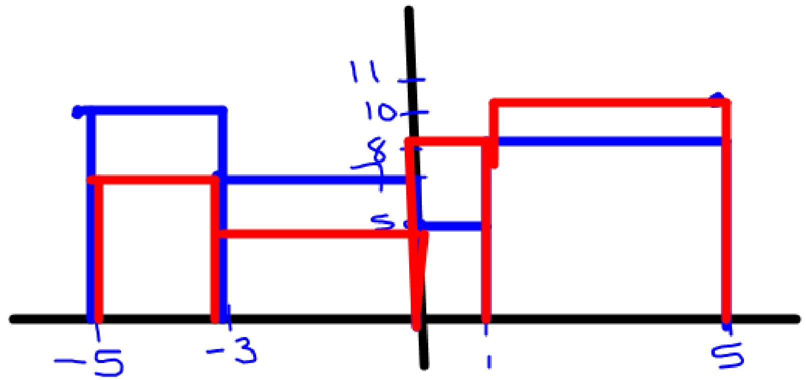
$\text{AREA} \approx 1\left[f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) \right]$

$\approx 1\left[0 + -4 + 0 + 6 + 8 + 0 \right]$

≈ 10

<u>Example 5:</u> The following table provides value for the function $f(x)$ on the interval [-5, 5]. Using the information available, approximate the area under the curve using the specified methods.

x	-5	-3	0	1	5
$f(x)$	10	7	5	8	11



a. Use left endpoints based off the table.

$$\text{Area} \approx 2\cdot f(-5) + 3\cdot f(-3) + 1\cdot f(0) + 4\cdot f(1)$$

$$20 + 21 + 5 + 32$$

$$\boxed{78}$$

b. Use right endpoints based off the table.

$$\text{Area} \approx 2f(-3) + 3f(0) + 1f(1) + 4f(5)$$

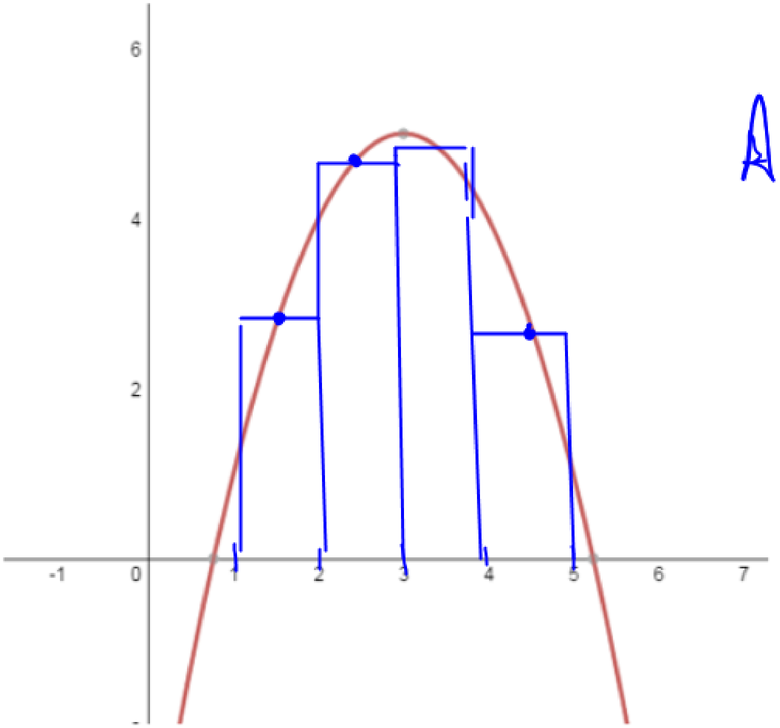
$$14 + 15 + 8 + 44$$

$$\boxed{81}$$

-When using left and right endpoint sums, the approximation can overestimate or underestimate the area under the curve by a significant amount. One way to counteract that error is to use a **Midpoint Riemann Sum.** This uses rectangle of a certain width and the midpoint of each interval.

<u>Example 6:</u> Approximate the area between the x-axis and the curve $y = -x^2 + 6x - 4$ on the interval $[1, 5]$ using a Midpoint Riemann Sum. Use 4 subintervals of equal width.

$\Delta x = \frac{5-1}{4} = 1$



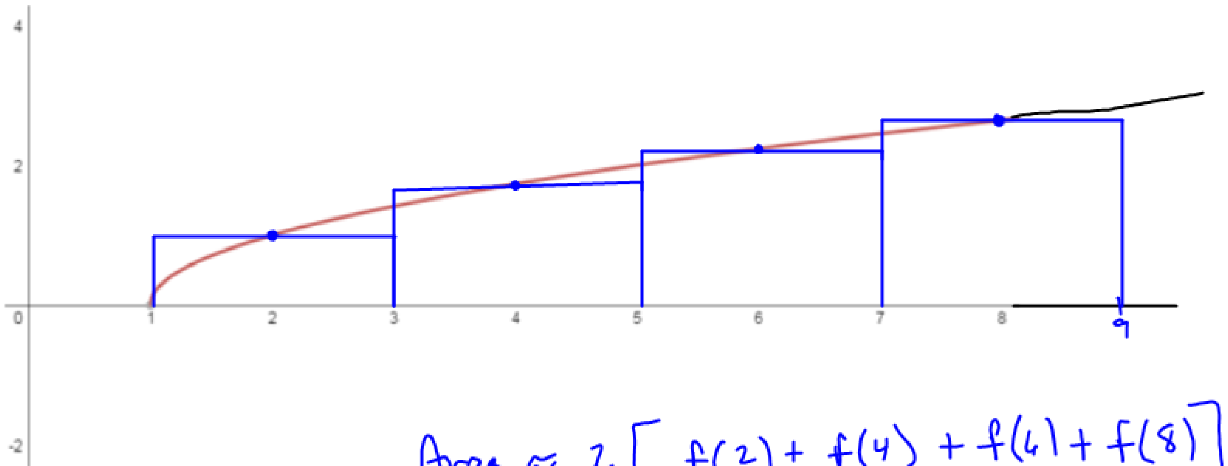
$$\text{Area} \approx 1\left[f\left(\tfrac{3}{2}\right) + f\left(\tfrac{5}{2}\right) + f\left(\tfrac{7}{2}\right) + f\left(\tfrac{9}{2}\right)\right]$$

$$\approx 1\left[2.75 + 4.75 + 4.75 + 2.75\right]$$

$$\approx 15$$

This also can help eliminate the issues with left and right hand Riemann sums where the last rectangle considered has a height of 0.

Example 7: Approximate the area between the x-axis and the curve $y = \sqrt{x-1}$ on the interval $[1, 9]$ using a Midpoint Riemann Sum. Use 4 subintervals of equal width.



$$\text{Area} \approx 2\left[f(2) + f(4) + f(6) + f(8)\right]$$

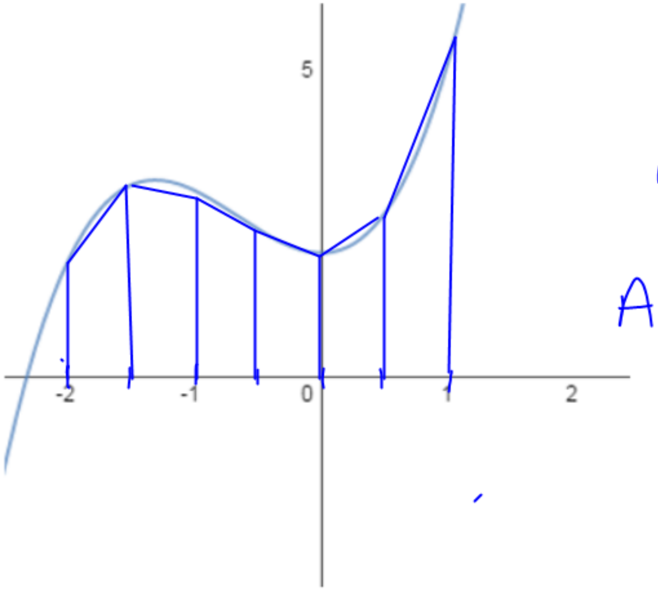
$$\approx 2\left[1 + \sqrt{3} + \sqrt{5} + \sqrt{7}\right]$$

$$\approx 15.228$$

-Another method that is used to approximate areas under curves is the **Trapezoidal Rule.** Using this, we still divide the interval into subintervals. We can create trapezoids using both endpoints of the interval and secant segments.

Example 8: Use the trapezoidal rule to find the approximate area between the x-axis and the curve $y = x^3 + 2x^2 + 2$ on the interval [-2, 1]. Use 6 subintervals of equal width.

$\Delta x = \frac{1 - -2}{6} = \frac{1}{2}$



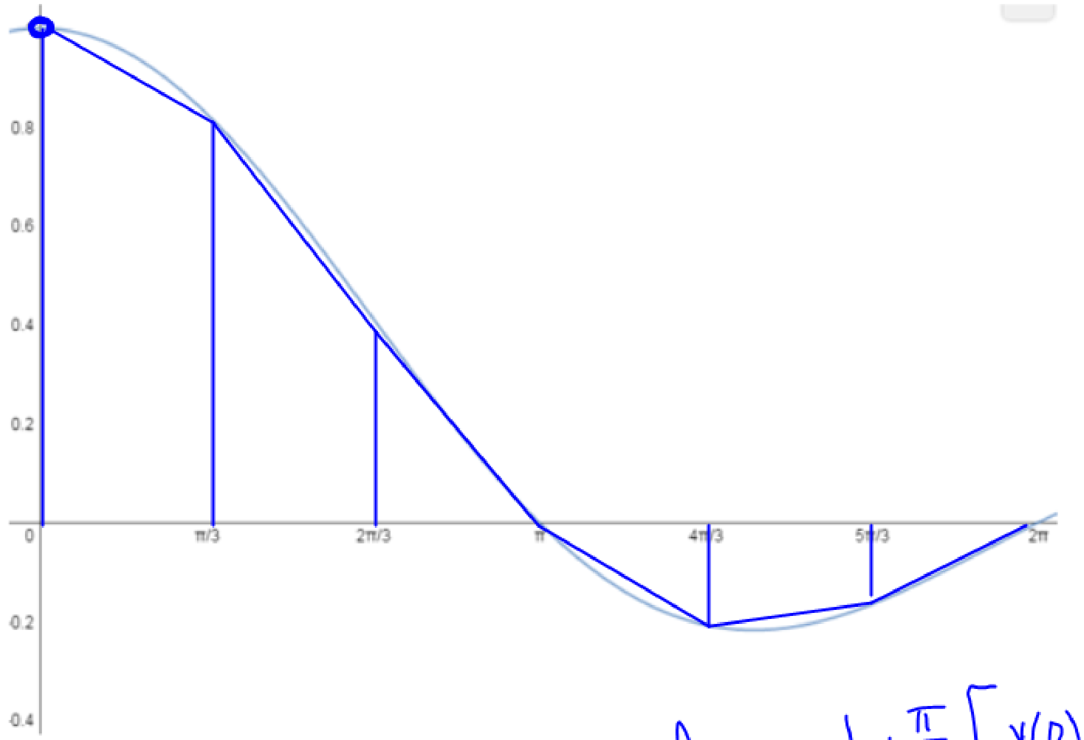
Δx

$\text{AREA} = \frac{1}{2} \cdot \frac{1}{2} \left[f(-2) + f(-\frac{3}{2}) + f(-\frac{3}{2}) + f(-1) + \ldots \right]$

$\text{AREA} \approx \frac{1}{2} \cdot \frac{1}{2} \left[f(-2) + 2f(-\frac{3}{2}) + 2f(-1) + 2f(-\frac{1}{2}) + 2f(0) + 2f(\frac{1}{2}) + f(1) \right]$

<u>Example 9:</u> Use the trapezoidal use to find the approximate area between the x-axis and the curve $y = \frac{\sin x}{x}$ on the interval $[0, 2\pi]$. Use 6 subintervals of equal width.

$\Delta x = \frac{2\pi - 0}{6} = \frac{\pi}{3}$



$$\text{AREA} \approx \frac{1}{2} \cdot \frac{\pi}{3}\left[\underbrace{y(0)}_{} + 2\cdot y\left(\frac{\pi}{3}\right) + 2\cdot y\left(\frac{2\pi}{3}\right) + 2\cdot y(\pi) + 2\cdot y\left(\frac{4\pi}{3}\right) + 2\cdot y\left(\frac{5\pi}{3}\right) + y(2\pi)\right]$$

$$y = \frac{\sin x}{x}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$