Answers to Worksheet on Parametrics and Calculus

1. $\frac{dy}{dx}=\frac{2t+6}{2t}=1+\frac{3}{t};\quad \frac{d^2y}{dx^2}=\frac{-\frac{3}{t^2}}{2t}=-\frac{3}{2t^3}$

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2. $\frac{dy}{dt}=3t-1;\quad \frac{d^2y}{dx^2}=\frac{3}{2t}$

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3. $\frac{dy}{dx}=\frac{6t+2}{\frac{1}{2}t^{-\frac{1}{2}}}=12t^{\frac{3}{2}}+4t^{\frac{1}{2}};\quad \frac{d^2y}{dx^2}=\frac{18t^{\frac{1}{2}}+2t^{-\frac{1}{2}}}{\frac{1}{2}t^{-\frac{1}{2}}}=36t+4$

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4. $\frac{dy}{dx}=\frac{2t+1}{\frac{1}{t}}=2t^2+t;\quad \frac{d^2y}{dx^2}=\frac{4t+1}{\frac{1}{t}}=4t^2+t$

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5. $\frac{dy}{dx}=\frac{-4\sin t}{3\cos t}=-\frac{4}{3}\tan t;\quad \frac{d^2y}{dx^2}=\frac{-\frac{4}{3}\sec^2 t}{3\cos t}=-\frac{4}{9}\sec^3 t$

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6. (a) $\frac{dy}{dx}=\frac{3t^2-2t}{2t+1}$

(b) When $t=1$, $\frac{dy}{dx}=\frac{3\cdot 1^2-2\cdot 1}{2\cdot 1+1}=\frac{8}{5}$, $x=5$, $y=4$ so the tangent line equation is

$y-4=\frac{8}{5}(x-5)$

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7. (a) $\frac{dy}{dx}=\frac{3\cos t}{-2\sin t}=-\frac{3}{2}\cot t$

(b) When $t=\frac{\pi}{4}$, $\frac{dy}{dx}=-\frac{3}{2}\cot\frac{\pi}{4}=-\frac{3}{2}$, $x=\sqrt{2}$, $y=\frac{3\sqrt{2}}{2}$ so the tangent line equation is

$y-\frac{3\sqrt{2}}{2}=-\frac{3}{2}\left(x-\sqrt{2}\right)$

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8. (a) $\frac{dy}{dx}=\frac{2t-4}{1}$

(b) A horizontal tangent occurs when $\frac{dy}{dt}=0$ and $\frac{dx}{dt}\neq 0$ so a horizontal tangent occurs when $2t-4=0$ which is at $t=2$. When $t=2$, $x=7$ and $y=-4$ so a horizontal tangent occurs at the point $(7,-4)$. A vertical tangent occurs when $\frac{dx}{dt}=0$ and $\frac{dy}{dt}\neq 0$

Since $1\neq 0$, there is no point of vertical tangency on this curve.

9. (a) $\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t - 1}$

(b) A horizontal tangent occurs when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$ so a horizontal tangent occurs when $3t^2 - 3 = 0$ which is at $t = \pm 1$. When $t = 1$, $x = 1$ and $y = -2$, and when $t = -1$, $x = 3$ and $y = 2$ so a horizontal tangent occurs at the points $(1, -2)$ and $(3, 2)$

A vertical tangent occurs when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$ so a vertical tangent occurs when $2t - 1 = 0$ so $t = \dfrac{1}{2}$. When $t = \dfrac{1}{2}$, $x = \dfrac{3}{4}$ and $y = -\dfrac{11}{8}$ so a vertical tangent occurs at the point $\left(\dfrac{3}{4}, -\dfrac{11}{8}\right)$.

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10. (a) $\dfrac{dy}{dx} = \dfrac{4\cos t}{-2\sin t}$

(b) A horizontal tangent occurs when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$ so a horizontal tangent occurs when $4\cos t = 0$ which is at $t = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$. When $t = \dfrac{\pi}{2}$, $x = 3$ and $y = 3$, and when $t = \dfrac{3\pi}{2}$, $x = 3$ and $y = -5$ so a horizontal tangent occurs at the points $(3, 3)$ and $(3, -5)$.

A vertical tangent occurs when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$ so a vertical tangent occurs when $-2\sin t = 0$ so $t = 0$ and π. When $t = 0$, $x = 5$ and $y = -1$ and when $t = \pi$, $x = 1$ and $y = -1$ so a vertical tangent occurs at the points $(5, -1)$ and $(1, -1)$.

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11. **a** $x = 0 \Rightarrow t = 2$

$x = 2 \Rightarrow t = 3$

b area $= \int_0^2 y \, dx$

$x = 2t - 4 \quad \therefore \dfrac{dx}{dt} = 2$

$\therefore$ area $= \int_2^3 \dfrac{1}{t} \times 2 \, dt$

$= \int_2^3 \dfrac{2}{t} \, dt$

c $= [2 \ln |t|]_2^3$

$= 2 \ln 3 - 2 \ln 2$

$= 2 \ln \frac{3}{2}$

d $t = \dfrac{x+4}{2}$

$\therefore y = \dfrac{2}{x+4}$

$\therefore$ area $= \int_0^2 \dfrac{2}{x+4} \, dx$

$= [2 \ln |x+4|]_0^2$

$= 2 \ln 6 - 2 \ln 4$

$= 2 \ln \frac{3}{2}$

12. **a** $x = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

for $y > 0$, $\theta = \frac{\pi}{2}$ at A

$y = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi$

for $x > 0$, $\theta = 0$ at B

b $x = 4\cos\theta \;\therefore\; \frac{dx}{d\theta} = -4\sin\theta$

$\therefore$ area $= \int_{\frac{\pi}{2}}^{0} 2\sin\theta \times -4\sin\theta \; d\theta$

$= \int_{0}^{\frac{\pi}{2}} 8\sin^2\theta \; d\theta$

c shaded area $= \int_{0}^{\frac{\pi}{2}} (4 - 4\cos 2\theta) \; d\theta$

$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{2}}$

$= (2\pi - 0) - (0 - 0)$

$= 2\pi$

area of ellipse $= 4 \times 2\pi$

$= 8\pi$

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13. **a** $y = 0 \Rightarrow \sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}$

$x = 2\sin t \;\therefore\; \frac{dx}{dt} = 2\cos t$

area above x-axis

$= \int_{0}^{\frac{\pi}{2}} 5\sin 2t \times 2\cos t \; dt$

$= \int_{0}^{\frac{\pi}{2}} 10\sin 2t\cos t \; dt$

area enclosed by curve

$= 2\int_{0}^{\frac{\pi}{2}} 10\sin 2t\cos t \; dt$

$= \int_{0}^{\frac{\pi}{2}} 20\sin 2t\cos t \; dt$

b $= 40\int_{0}^{\frac{\pi}{2}} \sin t\cos^2 t \; dt$

$= -40\int_{0}^{\frac{\pi}{2}} (-\sin t)\cos^2 t \; dt$

$= -40[\frac{1}{3}\cos^3 t]_0^{\frac{\pi}{2}}$

$= -\frac{40}{3}(0 - 1)$

$= 13\frac{1}{3}$